

## Fibred torti-rational knots

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### ABSTRACT

A torti-rational knot, denoted by  $K(2\alpha, \beta|r)$ , is a knot obtained from the 2-bridge link  $B(2\alpha, \beta)$  by applying Dehn twists an arbitrary number of times,  $r$ , along one component of  $B(2\alpha, \beta)$ . We determine the genus of  $K(2\alpha, \beta|r)$  and solve a question of when  $K(2\alpha, \beta|r)$  is fibred. In most cases, the Alexander polynomials determine the genus and fibredness of these knots. We develop both algebraic and geometric techniques to describe the genus and fibredness by means of continued fraction expansions of  $\beta/2\alpha$ . Then, we explicitly construct minimal genus Seifert surfaces. As an application, we solve the same question for the satellite knots of tunnel number one.

*Keywords:* Fibred knot, 2-bridge knot, satellite knot, tunnel number of knots, genus of knots, Alexander polynomial

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### 1. Introduction

A torti-rational knot <sup>a</sup>, denoted by  $K(2\alpha, \beta|r)$ , is a knot obtained from the 2-bridge link  $B(2\alpha, \beta)$  by applying Dehn twists an arbitrary number of times,  $r$ , along one component of  $B(2\alpha, \beta)$ . (For the precise definition, see Section 2.)

Torti-rational knots have occasionally appeared in literatures of knot theory. For example, twist knots are torti-rational knots. By [15], we know when  $K(2\alpha, \beta|r)$  is unknotted (see Proposition 6.6). A torti-rational knot is a  $g1$ - $b1$  knot (i.e., admits a genus-one bridge-one decomposition), and hence has tunnel number one. In 1991, Morimoto and Sakuma [17] proved that for a satellite knot of tunnel number one, the companion knot is a torus knot  $T(p, q)$  and the pattern knot is a torti-rational knot  $K(2\alpha, \beta|pq)$ , for some  $p, q$ , and  $\alpha, \beta$ . Then, Goda and Teragaito [8] determined which of such satellite knots of tunnel number one are of genus one.

In this paper, we study torti-rational knots systematically and completely determine the genus of any torti-rational knot and solve a question of when it is

<sup>a</sup> This naming is due to Lee Rudolph.

fibred.

In fact, we prove the following:

**Theorem A.** (Theorems 2.1 and 2.2) *Let  $B(2\alpha, \beta)$  be an oriented 2-bridge link, with linking number  $\ell$ . Let  $K = K(2\alpha, \beta|r)$  be a torti-rational knot. Suppose  $\ell \neq 0$ . (1) The genus of  $K$  is exactly half of the degree of the Alexander polynomial  $\Delta_K(t)$ . (2)  $K$  is fibred if (and only if)  $\Delta_K(t)$  is monic (i.e., the leading coefficient is  $\pm 1$ ).*

See Theorems 6.1 and 9.1 for a practical method for determination.

If  $\ell = 0$ , Theorem A does not hold true. For this case, the genus and the characterization of a fibred torti-rational knot are stated as follows:

**Theorem B.** (Theorems 2.3 and 2.4) *Suppose  $\ell = 0$ , Let  $[2c_1, 2c_2, \dots, 2c_m]$  be the continued fraction of  $\frac{\beta}{2\alpha}$ .*

(1) *For any  $r \neq 0$ ,  $g(K(2\alpha, \beta|r)) = \frac{1}{2} \sum_{i: \text{ odd}} |c_i|$ .*

(2) (a) *If  $|r| > 1$ , then  $K(2\alpha, \beta|r)$  is not fibred. (b) Suppose  $r = \pm 1$ . Then  $K(2\alpha, \beta|r)$  is fibred if and only if  $\frac{\beta}{2\alpha}$  has the continued fraction of the following special form:  $\frac{\beta}{2\alpha} = \pm[2a_1, 2, 2a_2, 2, \dots, 2a_p, \pm 2, -2a'_1, -2, -2a'_2, -2, \dots, -2a'_q]$ , where  $2\alpha > \beta > -2\alpha$ ,  $a_i, a'_j > 0$  and  $\sum_{i=1}^p a_i = \sum_{j=1}^q a'_j$ .*

See Section 2, for our convention of continued fractions.

To prove these theorems, we construct explicitly a minimal genus Seifert surface for  $K$  and determine whether or not it is a fibre surface for  $K$ . Proofs of these theorems will be given in Sections 10 and 11.

This work is a part of our project to determine the genus and fibredness of double torus knots (i.e., knots embedded in a standard closed surface of genus 2). See [10] for a relevant work. Double torus knots are classified into five types (Type  $(1, 1)$ ,  $(1, 2)$ ,  $(1, 3)$ ,  $(2, 3)$  and  $(3, 3)$ ). In [10], we settled the problem for all double torus knots of type  $(1, 1)$ . A  $g1$ -b1 knot can be presented as a double torus knot of type either  $(1, 2)$ ,  $(2, 2)$  or  $(2, 3)$ . In this paper, we settle the problem for the  $g1$ -b1 knots presented as of type  $(1, 2)$ . As an application of our study, we determine the genus and the fibredness problem for the satellite knots of tunnel number one. In fact, we show that a similar theorem to Theorem A holds true for satellite knots in a slightly wider class. The precise statements can be found in Section 13.

Recently, Goda, Hayashi and Song [7] study torti-rational knots with a different motivation. Their approach is completely different from ours.

This paper is organized as follows. In Section 2, we give precise statements of our main theorems (Theorems 2.1 - 2.4). In Section 3, we first introduce several notions needed in this paper, such as *graphs of continued fractions*, *dual graphs*. Then we prove that for our study of  $K(2\alpha, \beta|r)$ , we may assume  $\ell \geq 0$  and  $r > 0$ , where  $\ell$  is the linking number of  $B(2\alpha, \beta)$ . This restriction simplifies considerably the proofs of our main theorems. At the end of Section 3, we construct a spanning disk of a nice form for one component of the 2-bridge link. In Sections 4 and 5, we study the Alexander polynomial of various knots: In Section 4, we determine the Alexander polynomials of  $K(2\alpha, \beta|r)$ . In Section 5, we prove one subtle property of

the (2-variable) Alexander polynomial of  $B(2\alpha, \beta)$ . The determination of the degree of the Alexander polynomials of  $K(2\alpha, \beta|r)$  depends on this property. Sections 6 is devoted to characterizing the monic Alexander polynomial of a knot  $K(2\alpha, \beta|r)$ : First, we deal with knots  $K(2\alpha, \beta|r)$  for the case  $\ell > 0$ , and characterize the monic Alexander polynomials in terms of a continued fraction of  $\beta/2\alpha$  using the formulae given in Section 5. In particular, we give an equivalent algebraic condition for Theorem 2.2 (Theorem 6.1). However, for the case  $\ell = 0$ , the monic Alexander polynomials of  $K(2\alpha, \beta|r)$  cannot be characterized by the continued fractions. This case is considered in the rest of Section 6, and the characterization will be done using a geometric interpretation of the Alexander polynomials of  $K(2\alpha, \beta|r)$ . In Section 7, we construct explicitly a Seifert surface for  $K(2\alpha, \beta|r)$ , which in most cases is of minimal genus. In Section 8, we prove Theorem 2.1. In this case, some of the surfaces constructed in Section 7 are not of minimal genus, but we obtain minimal genus surfaces after explicitly compressing them. In Section 9, we prove Theorem 2.2. In Sections 10 and 11, we prove Theorems 2.3 and 2.4. Various examples that illustrate our main theorems are discussed in Section 12. In section 13, we consider the satellite knot with fibred companion and  $K(2\alpha, \beta|r)$ ,  $r \neq 0$ , as a pattern, and prove an analogous theorem to Theorem A. In the final section, Section 14, we determine the genus one knots in our family of knots  $K(2\alpha, \beta|r)$ . In particular, we find satellite knots among them, and hence give a negative answer to the problem posed in [1].

After is this paper was completed, D. Silver pointed out that Theorem 5.5 in this paper makes it possible to prove Theorem A algebraically using Brown's graphs in [2] (without explicit construction of minimal genus Seifert surfaces). However, Brown's method does not work for proving Theorem B.

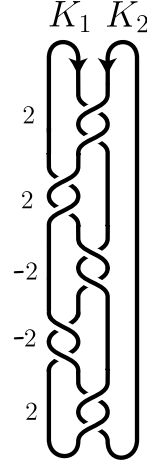
## 2. Statements of main theorems

We begin with an (even) continued fraction of a rational number  $\frac{\beta}{2\alpha}$ ,  $0 < \beta < 2\alpha$ , and  $\gcd(2\alpha, \beta) = 1$ . The (unique) continued fraction of

$$\frac{\beta}{2\alpha} = \cfrac{1}{2c_1 - \cfrac{1}{2c_2 - \cfrac{1}{2c_3 - \cfrac{1}{\ddots - \cfrac{1}{2c_{m-1} - \cfrac{1}{2c_m}}}}}},$$

where  $c_i \neq 0$ , is denoted by  $\frac{\beta}{2\alpha} = [2c_1, 2c_2, \dots, 2c_m]$  or  $[[c_1, c_2, \dots, c_m]]$ . Note that  $m$  is odd. Throughout this paper, we consider only even continued fraction expansions, and hence omit the word 'even'. Now, using the continued fraction of  $\frac{\beta}{2\alpha}$ , we can obtain a diagram of an oriented 2-bridge link  $B(2\alpha, \beta)$  as follows.

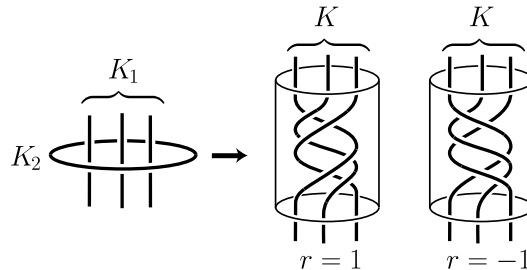
Let  $\sigma_1 = \begin{array}{c} \diagup \quad \diagdown \\ | \end{array}$  and  $\sigma_2 = \begin{array}{c} | \quad \diagup \quad \diagdown \end{array}$  be Artin's generators of the 3-braid group. First construct a 3-braid  $\gamma = \sigma_2^{2c_1} \sigma_1^{2c_2} \sigma_2^{2c_3} \cdots \sigma_2^{2c_m}$ . Close  $\gamma$  by joining the first and second strings (at the both ends) and then join the top and bottom of the third string by a simple arc as in Figure 2.1. We give downward orientation to the second and third strings. Figure 2.1 shows the (oriented) 2-bridge link obtained from the continued fraction  $\frac{21}{34} = [2, 2, -2, -2, 2] = [[1, 1, -1, -1, 1]]$ . Now an oriented 2-bridge link  $B(2\alpha, \beta)$  consists of two unknotted knots  $K_1$  and  $K_2$ , where  $K_2$  is formed from the third and fourth strings.

Figure 2.1:  $S(34, 21)$ 

**Note.** Our convention for the orientation of a 2-bridge link is not standard, but is used for the convenience in utilizing the 2-variable Alexander polynomials. (Usually we reverse the orientation of one component so that the 2-bridge link is fibred if and only if all the entries of the even continued fraction are  $\pm 2$ .)

Since  $K_2$  is unknotted,  $K_1$  can be considered as a knot in an unknotted solid torus  $V$  and  $K_2$  is a meridian of  $V$ . Then by applying Dehn twists along  $K_2$  in an arbitrary number of times, say  $r$ , we obtain a new knot  $K$  from  $K_1$ . We denote this knot  $K$  by  $K(2\alpha, \beta|r)$ , or simply  $K(r)$ .

More precisely, one Dehn twist along  $K_2$  is the operation that replaces the part of  $K_1$  in a cylinder by the braid  $(\sigma_1 \sigma_2 \cdots \sigma_{k-1})^k$ , where  $k$  is the wrapping number. See Figure 2.2. (Since  $B(2\alpha, \beta)$  is symmetric,  $K_1$  and  $K_2$  can be interchanged, and hence this notation is justified.)

Figure 2.2: Dehn twists along  $K_2$ 

We note that if  $r = 0$ , then  $K(2\alpha, \beta|r)$  is unknotted for any  $\alpha, \beta$ , and henceforth we assume  $r \neq 0$  unless otherwise specified.

Now, given an oriented 2-bridge link  $B(2\alpha, \beta)$ , let  $\ell = \ell k(K_1, K_2)$  be the linking number between  $K_1$  and  $K_2$  which, for simplicity, is denoted by  $\ell k B(2\alpha, \beta)$ .

Our main theorems in this paper are the following four theorems:

**Theorem 2.1.** *Suppose  $\ell \neq 0$ . Then the genus of  $K = K(2\alpha, \beta|r)$  is half of the degree of its Alexander polynomial  $\Delta_K(t)$ . Namely we have  $g(K) = \frac{1}{2} \deg \Delta_K(t)$ .*

**Theorem 2.2.** *Suppose  $\ell \neq 0$ . Then  $K = K(2\alpha, \beta|r)$  is a fibred knot if (and only if)  $\Delta_K(t)$  is monic, i.e.,  $\Delta_K(0) = \pm 1$ .*

Theorem 2.2 is divided into two parts: Theorem 6.1 is the algebraic part, where we determine when  $\Delta_K(t)$  is monic in terms of continued fractions, and Theorem 9.1 is the geometric part, where we show the fibredness, by actually constructing fibre surfaces using the continued fractions.

**Theorem 2.3.** *Suppose  $\ell = 0$ . Let  $[2c_1, 2c_2, \dots, 2c_m]$  be the continued fraction of  $\frac{\beta}{2\alpha}$ . Then for any  $r \neq 0$ ,  $g(K(2\alpha, \beta|r)) = \frac{1}{2} \sum_{i: \text{ odd}} |c_i|$ .*

**Theorem 2.4.** *Suppose  $\ell = 0$ . (a) If  $|r| > 1$ , then  $K(2\alpha, \beta|r)$  is not fibred. (b) Suppose  $r = \pm 1$ . Then  $K(2\alpha, \beta|r)$  is fibred if and only if  $\frac{\beta}{2\alpha}$  has the continued fraction of the following special form:  
 $\frac{\beta}{2\alpha} = \pm[2a_1, 2, 2a_2, 2, \dots, 2a_p, \pm 2, -2a'_1, -2, -2a'_2, -2, \dots, -2a'_q]$ , where  $2\alpha > \beta > -2\alpha$ ,  $a_i, a'_j > 0$  and  $\sum_{i=1}^p a_i = \sum_{j=1}^q a'_j$ .*

In Theorem 2.4, we have non-fibred knots  $K$  such that  $\Delta_K(t)$  are monic and  $\deg \Delta_K(t) = 2g(K)$ .

### 3. Preliminaries

In this section, we first introduce two fundamental concepts, a *graph of a continued fraction* and the *dual graph*, which play a key role throughout this paper. Next, in Subsection 3.4, we show that we can assume  $\ell k B(2\alpha, \beta) \geq 0$  and  $r > 0$  without loss of generality. This assumption is very important to simplify the proof of our main theorem. In the last Subsection (Subsection 3.5), we introduce the concept of the primitive spanning disk for  $K_1$ , which is the first step of constructing a minimal genus Seifert surface for  $K(2\alpha, \beta|r)$ .

#### 3.1. Modified continued fractions and their graphs.

Let  $S = [[c_1, c_2, \dots, c_{2k+1}]]$  be the continued fraction of  $\frac{\beta}{2\alpha}$ , where  $-2\alpha < \beta < 2\alpha$  and  $\gcd(2\alpha, \beta) = 1$ . The *length* of  $S$  is defined as  $2k + 1$ . To define the dual of  $S$ , we need to extend  $S$  slightly to  $S^*$ , called the modified form of  $S$ . We will see that  $S$  and  $S^*$  correspond to the same rational number.

**Definition 3.1.** Let  $S = [[c_1, c_2, \dots, c_{2k+1}]]$ . Then we obtain a continued fraction  $S^*$  by thoroughly repeating the following and call it *modified form of  $S$* .

- (1) If  $c_{2i+1} > 1$ ,  $0 \leq i \leq k$ , then  $c_{2i+1}$  is replaced by the new sequence of length  $2c_{2i+1} - 1$ ,  $(1, 0, 1, 0, \dots, 0, 1)$ , and

(2) if  $c_{2i+1} < -1$ ,  $0 \leq i \leq k$ , then  $c_{2i+1}$  is replaced by the new sequence of length  $2|c_{i+1}| - 1$ ,  $(-1, 0, -1, 0, \dots, 0, -1)$ . Note that the length of  $S^*$  is

$$\sum_{i=0}^k (2|c_{2i+1}| - 2) + 2k + 1 = \sum_{i=0}^k 2|c_{2i+1}| - 1.$$

The original continued fraction  $S$  may be called the *standard* continued fraction of  $\beta/2\alpha$ , which does not contain entries 0.

The modified form of  $\beta/2\alpha$  is of the form:

$$[2u_1, 2v_1, 2u_2, 2v_2, \dots, 2u_d, 2v_d, 2u_{d+1}], \quad (3.1)$$

where  $u_i = +1$  or  $-1$ , for  $1 \leq i \leq d+1$ , and  $v_i$ , ( $1 \leq i \leq d$ ) are arbitrary, including 0.

Now, given the continued fraction  $S$  of  $\beta/2\alpha$ , consider the modified form  $S^*$  for  $S$  of the form (3.1).

**Definition 3.2.** The *graph*  $G(S^*)$  of  $S^*$ , (or the graph  $G(S)$  of  $S$ ), is a plane graph in  $\mathbb{R}^2$ , consisting of  $d+2$  vertices  $V_0, V_1, \dots, V_{d+1}$  and  $d+1$  line segments  $E_k$ , ( $1 \leq k \leq d+1$ ) joining two vertices  $V_{k-1}$  and  $V_k$ , where  $V_0 = (0, 0)$  and  $V_i = (i, \sum_{j=1}^i u_j)$ , for  $1 \leq i \leq d+1$ . The graph is a *weighted graph*, when the weight of  $V_i$ , ( $1 \leq i \leq d$ ) is defined as  $2v_i$ . The weights of both  $V_0$  and  $V_{d+1}$  are 0.

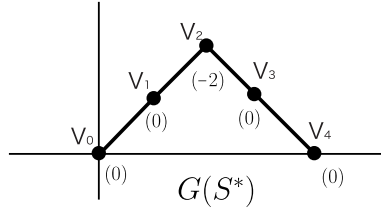


Figure 3.1: Graph  $G(S^*)$  for  $S^* = [2, 0, 2, -2, -2, 0, -2]$

**Example 3.3.** Let  $S^* = [2, 0, 2, -2, -2, 0, -2]$ . Then  $G(S^*)$  is depicted in Figure 3.1. The weight of  $V_i$  is denoted by  $(m)$  near  $V_i$ .

The following is immediate from the diagram of  $B(2\alpha, \beta)$  (Figure 2.1).

**Proposition 3.4.** The  $y$ -coordinate of the last vertex  $V_{d+1}$  gives the linking number  $\ell = \ell k B(2\alpha, \beta)$ . Namely,  $\ell = \sum_{i=1}^{d+1} u_i$ .

### 3.2. Dual graphs and dual continued fractions.

For a continued fraction  $S$ , we define the dual  $\tilde{S}$  to  $S$  and the dual graph  $\tilde{G}$  to a graph  $G(S)$ . Then we have the following theorem, proved in Subsection 3.5:

**Theorem 3.5.** Let  $S$  be the continued fraction of  $\beta/2\alpha$ . Then the dual  $\tilde{S}$  of  $S$  is the continued fraction of  $(2\alpha - \beta)/2\alpha$  (resp.  $(-2\alpha - \beta)/2\alpha$ ), if  $\beta > 0$  (resp.  $\beta < 0$ ).

**Definition 3.6.** The *dual*  $\tilde{G}$  to the graph  $G(S)$  is defined as follows. The underlying graph of  $\tilde{G}$  is exactly the same as that of  $G(S)$ , but the weight  $\tilde{w}(V_i)$  is given as follows;

- (1) If  $V_i$  is a local maximal or local minimal vertex (including the ends of  $G(S)$ ), then  $\tilde{w}(V_i) = -w(V_i)$ , and
- (2) for the other vertices,  $\tilde{w}(V_i) = 2\varepsilon_i - w(V_i)$ , where  $\varepsilon_i$  is the sign of  $u_i$ , i.e.,  $\varepsilon_i = u_i/|u_i|$

The dual  $\tilde{S}^*$  to the modified form  $S^*$  is defined to be the modified form of the continued fraction represented by the dual graph  $\tilde{G}$ . The dual  $\tilde{S}$  of  $S$  is the standard continued fraction obtained from  $\tilde{S}^*$ .

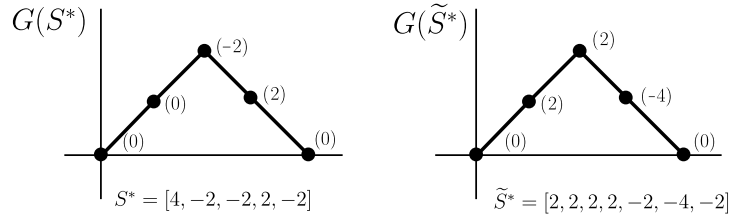


Figure 3.2: The graphs for  $S^*$  and  $\tilde{S}^*$

**Example 3.7.** Let  $S = [[2, -1, -1, 1, -1]]$ .

Then  $S^* = [[1, 0, 1, -1, -1, 1, -1]] = [4, -2, -2, 2, -2]$ .

Thus  $\tilde{S}^* = [[1, 1, 1, 1, -1, -2, -1]] = [2, 2, 2, 2, -2, -4, -2] = \tilde{S}$ .

In the following, we give an alternative formulation of  $\tilde{S}$ , the dual of  $S$ . Given the continued fraction of  $\beta/2\alpha$ ,

$$[[c_1, c_2, \dots, c_{2d+1}]], c_i \neq 0, \quad (3.2)$$

consider the partial sequence of (3.2):

$$\{c_1, c_3, c_5, \dots, c_{2d+1}\}, \quad (3.3)$$

consisting of only  $c_{2i+1}, 0 \leq i \leq d$ .

In this sequence, for convenience, write  $-c_i$ , where  $c_i < 0$ , so that we may assume that if  $i$  is odd, then  $c_i$  is always positive. Thus, the sequence (3.3) is divided into several ‘positive’ or ‘negative’ sub-sequences: Therefore we can write,  $[[c_1, c_2, \dots, c_{2d+1}]] = [[a_1, b_1, a_2, \dots, a_p, b_p, -a_{p+1}, -b_{p+1}, \dots, -a_r, b_r, a_{r+1}, b_{r+1}, \dots]]$ , where  $a_i > 0$  for all  $i$ . Note that  $a_i = c_{2i-1}$  or  $-c_{2i-1}, i = 1, 2, \dots$  and  $b_j = c_{2j}, j = 1, 2, \dots$

We call a sequence of the form  $[[a_1, b_1, a_2, b_2, \dots, a_k, b_k, a_{k+1}]]$  (or  $[[a_1, b_1, a_2, b_2, \dots, a_k, b_k, a_{k+1}]]$ ) a *positive* (or *negative*) sequence, where  $a_i > 0, 1 \leq i \leq k+1$ , but  $b_j, (1 \leq j \leq k)$  are arbitrary ( $\neq 0$ ). We denote by

$P_i$  (resp.  $Q_i$ ) a positive (resp. negative) subsequence.

**Example 3.8.**

$$\begin{aligned} & [[1, 1, 2, -1, 1, -1, -2, 1, -2, -1, 2, 1, 2, 1, -2, -1, -2]] \\ &= [[\underbrace{1, 1, 2, -1, 1, -1}_{P_1}, \underbrace{-2, -(-1), -2, -1}_{Q_1}, \underbrace{2, 1, 2, 1}_{P_2}, \underbrace{-2, -1, -2}_{Q_2}]] \\ &= [[P_1, -1, Q_1, -1, P_2, 1, Q_2]] \end{aligned}$$

Thus, the sequence (3.2) can be written as

$$[c_1, c_2, c_3, \dots, c_{2d+1}] = \{P_1, d_1, Q_1, e_1, P_2, d_2, Q_2, e_2, P_3, \dots\}, \quad (3.4)$$

where  $d_i, e_j$  are some  $c_{2k}$ .

This form (3.4) is called the *canonical decomposition* of the continued fraction of  $\beta/2\alpha$ .

**Remark 3.9.** If  $\beta > 0$ , then the first entry  $c_1 > 0$ , but if  $\beta < 0$ , then  $c_1 < 0$ , and hence, the canonical decomposition begins with  $Q_1$  (not a positive sequence  $P_1$  and  $d_1$  is missing). However, since this does not change our argument, we may assume in general that  $c_1 > 0$ .

Now, the *dual continued fraction* of (3.2) is reformulated as follows.

Let  $S = \{P_1, d_1, Q_1, e_1, P_2, \dots\}$  be the canonical decomposition of  $[c_1, c_2, \dots, c_{2d+1}]$ .

First the *dual of a positive sequence*  $P$  is obtained as follows.

Given  $P = [[a_1, b_1, a_2, b_2, \dots, a_m]]$ ,  $a_j > 0$ , consider the modified form  $P^*$  of  $P$

$$P^* = [[a_1^*, b_1^*, a_2^*, b_2^*, \dots, a_k^*, b_k^*, a_{k+1}^*]], \quad (3.5)$$

where  $a_j^* = 1$  ( $1 \leq j \leq k+1$ ) and  $b_j^*$ 's ( $1 \leq j \leq k$ ) are arbitrary including 0.

Then the dual of  $P^*$ , denoted by  $\tilde{P}^*$ , is  $\tilde{P}^* = [[\tilde{a}_1^*, \tilde{b}_1^*, \tilde{a}_2^*, \tilde{b}_2^*, \dots, \tilde{a}_{k+1}^*]]$ , where  $\tilde{a}_j^* = a_j^* = 1$ , for  $1 \leq j \leq k+1$  and  $\tilde{b}_j^* = 1 - b_j^*$  for  $1 \leq j \leq k$ .

The *dual*  $\tilde{P}$  of  $P$  is the standard form obtained from  $\tilde{P}^*$ .

For the negative sequence  $Q$ , apply the same operation for the positive sequence  $-Q$  to obtain the dual  $\widetilde{-Q}$  of  $-Q$ . Then the dual  $\tilde{Q}$  of  $Q$  is the negative sequence  $-(\widetilde{-Q})$ .

Finally, the *dual*  $\tilde{S}$  of  $S$  is  $\{\tilde{P}_1, -d_1, \tilde{Q}_1, -e_1, \tilde{P}_2, -d_2, \tilde{Q}_2, -e_2, \dots\}$ .

**Example 3.8 (continued)**

(1) Since  $P_1 = [[1, 1, 2, -1, 1]]$ ,  $P_1^* = [[1, 1, 1, 0, 1, -1, 1]]$ , and hence  $\tilde{P}_1^* = [[1, 0, 1, 1, 1, 2, 1]]$  and  $\tilde{P}_1 = [[2, 1, 1, 2, 1]]$ .

(2) Since  $P_2 = [[2, 1, 2]]$ ,  $P_2^* = [[1, 0, 1, 1, 1, 0, 1]]$ , and hence  $\tilde{P}_2^* = [[1, 1, 1, 0, 1, 1, 1]]$  and  $\tilde{P}_2 = [[1, 1, 2, 1, 1]]$ .

(3) Since  $Q_1 = [[-2, 1, -2]]$ ,  $-Q_1 = [[2, -1, 2]]$  and hence

$(-Q_1)^* = [[1, 0, 1, -1, 1, 0, 1]]$ ,  
 $-\tilde{Q}_1^* = [[1, 1, 1, 2, 1, 1, 1]] = -Q_1$ , so  $\tilde{Q}_1 = [[-1, -1, -1, -2, -1, -1, -1]]$



(4) Since  $Q_2 = [[-2, -1, -2]]$ ,  $-Q_2 = [[2, 1, 2]]$  and hence  $-Q_2^* = [[1, 1, 2, 1, 1]]$ , so  $\tilde{Q}_2 = [[-1, -1, -2, -1, -1]]$ . Thus,  $\tilde{S} = [[2, 1, 1, 2, 1, 1, -1, -1, -1, -2, -1, -1, -1, 1, 1, 1, 2, 1, 1, -1, -1, -1, -2, -1, -1]]$

### 3.3. Applications.

In this subsection, we study some of the invariants of a 2-bridge link  $B(2\alpha, \beta)$  deduced from  $S$  or its dual  $\tilde{S}$ .

The following three propositions show that  $S$  or  $\tilde{S}$  determines the degree of the Alexander polynomial  $\Delta_{B(2\alpha, \beta)}(x, y)$  of  $B(2\alpha, \beta)$ .

Let  $S = \{P_1, d_1, Q_1, e_1, P_2, \dots, P_m, d_m, Q_m\}$  be the canonical decomposition of the continued fraction of  $\beta/2\alpha$ . We write more precisely:

$$\begin{aligned} P_i &= [[a_{i,1}, b_{i,1}, a_{i,2}, b_{i,2}, \dots, b_{i,s_i}, a_{i,s_i+1}]], \text{ and} \\ Q_j &= [[-a'_{j,1}, -b'_{j,1}, -a'_{j,2}, -b'_{j,2}, \dots, -b'_{j,q_j}, -a'_{j,q_j+1}]] \end{aligned} \quad (3.6)$$

**Definition 3.10.** For  $1 \leq i, j \leq m$ , define

$$\begin{aligned} \rho_i &= |\{b_{i,\ell} | b_{i,\ell} = 1, 1 \leq \ell \leq s_i\}|, \\ \rho'_j &= |\{b'_{j,\ell} | b'_{j,\ell} = 1, 1 \leq \ell \leq q_j\}|, \text{ and} \\ \rho &= \rho(\beta/2\alpha) = \sum_{i=1}^m \rho_i + \sum_{j=1}^m \rho'_j. \end{aligned} \quad (3.7)$$

We call this  $\rho$  the *deficiency* (see Theorem 5.5 and Sections 6 and 8).

Further, we define;

$$\begin{aligned} \lambda_i &= \sum_{\ell=1}^{s_i+1} a_{i,\ell}, 1 \leq i \leq m \\ \lambda'_j &= \sum_{\ell=1}^{q_j+1} a'_{j,\ell}, 1 \leq j \leq m, \text{ and} \\ \lambda &= \sum_{i=1}^m \lambda_i + \sum_{j=1}^m \lambda'_j. \end{aligned} \quad (3.8)$$

Note that  $\lambda$  equals the number of edges in  $G(S)$ , which also equals the number of disks in Figure 3.4. This number is neatly evaluated by Kanenobu as follows :

**Proposition 3.11.** [12, (4.10)] Write  $\Delta_{B(2\alpha, \beta)}(x, y) = \sum_{0 \leq i, j} c_{i,j} x^i y^j \in \mathbb{Z}[x, y]$  in such a way that  $\min y\text{-deg } \Delta_{B(2\alpha, \beta)}(x, y) = \min\{j | c_{i,j} \neq 0\} = 0$ . Then  $\max y\text{-deg } \Delta_{B(2\alpha, \beta)}(x, y) = \max\{j | c_{i,j} \neq 0\} = \lambda - 1$ .

The following proposition shows that  $\lambda$  and  $\rho$  are related to the dual of  $S$ .

**Proposition 3.12.** Let  $\tilde{S}$  be the dual of  $S$ . Then the length of  $\tilde{S}$  is  $2(\lambda - \rho) - 1$ .

*Proof.* First consider the positive sequence  $P_i$ .

Let  $P_i = [2a_{i,1}, 2b_{i,1}, 2a_{i,2}, \dots, 2a_{i,s_i}, 2b_{i,s_i}, 2a_{i,s_i+1}]$ . Then the length of  $P_i$  is  $2s_i + 1$ . Now to get the dual, consider the modified form  $P_i^*$  of  $P_i$  that is of the form:

$$P_i^* = [2, 0, \underbrace{2, \dots, 0}_{2a_{i,1}-1}, 2b_{i,1}, \underbrace{2, 0, 2, \dots, 0}_{2a_{i,2}-1}, 2, 2b_{i,2}, \dots].$$

Then to obtain  $\tilde{P}_i$ , replace 0 in  $P_i^*$  by 2 and  $b_{i,r}$  by  $1 - b_{i,r}$ . Therefore, in  $\tilde{P}_i$ , 0 occurs exactly  $\rho_i$  times. Since the length of  $P_i^*$  is  $\sum_{r=1}^{s_i+1} (2a_{i,r} - 1) + s_i$ , the length of the dual  $\tilde{P}_i$  is

$$\begin{aligned} \sum_{r=1}^{s_i+1} (2a_{i,r} - 1) + s_i - 2\rho_i &= \sum_{r=1}^{s_i+1} 2a_{i,r} - (s_i + 1) + s_i - 2\rho_i \\ &= 2\lambda_i - 2\rho_i - 1. \end{aligned}$$

By the same reasoning, the length of the dual  $\tilde{Q}_j$  of  $Q_j$  is equal to  $2\lambda'_j - 2\rho'_j - 1$ .

Therefore, the length of the dual  $\tilde{S}$  of  $S$  is

$$\sum_{i=1}^m (2\lambda_i - 2\rho_i - 1) + \sum_{j=1}^m (2\lambda'_j - 2\rho'_j - 1) + (2m - 1) = 2\lambda - 2\rho - 1.$$

Note that Proposition 3.12 holds if  $Q_m = \phi$  (and hence  $d_m$  is missing) or  $P_1 = \phi$  (and hence  $d_1$  is missing). □

Combining Proposition 2 in [13] with Proposition 3.12 and Proposition 3.17 (in the next subsection), we obtain:

**Proposition 3.13.** *Let  $\Delta_{B(2\alpha, \beta)}(x, y)$  be the Alexander polynomial of a 2-bridge link  $B(2\alpha, \beta)$ . Then  $\Delta_{B(2\alpha, \beta)}(t, t)$  is a polynomial of degree  $2(\lambda - \rho - 1)$ .*

*Proof.* Let  $\Delta_B(t)$  be the reduced Alexander polynomial of  $B(2\alpha, \beta)$ . Then  $\Delta_B(t) = \Delta_{B(2\alpha, \beta)}(t, t)(1 - t)$ . Apply Proposition 2 in [13]. □

### 3.4. Reduction.

In this subsection, we justify the assumptions  $\ell \geq 0$  and  $r > 0$ . This restriction drastically simplifies the proofs of the main theorems.

For a knot  $K$ , we denote by  $\overline{K}$  the mirror image of  $K$ .

**Theorem 3.14.** *In studying the genera and fibredness of  $K(2\alpha, \beta|r)$  with  $r \neq 0$ , we may assume:*

(1)  $-2\alpha < \beta < 2\alpha$ , (2)  $\ell k B(2\alpha, \beta) \geq 0$ , and (3)  $r > 0$ .

*More precisely, we have the following: Suppose  $0 < \beta < 2\alpha$ . Then, for any  $r \neq 0$*

(I)  $K(2\alpha, \beta|r) = \overline{K(2\alpha, -\beta|-r)}$ .

(II)  $K(2\alpha, \beta|r) = \overline{K(2\alpha, 2\alpha - \beta|-r)}$  and  $K(2\alpha, -\beta|r) = \overline{K(2\alpha, -2\alpha + \beta|-r)}$ .

We remark the following: (i)  $\ell k B(2\alpha, \beta) = -\ell k B(2\alpha, -\beta)$ , (ii)  $\ell k B(2\alpha, \beta) = \ell k B(2\alpha, \pm 2\alpha - \beta)$ , (iii) if  $\ell B(2\alpha, \beta) = 0$ , then we may assume  $r > 0$  and  $\beta > 0$ .

**Example 3.15.** (1)  $K(4, -1|3) = \overline{K(4, 1|-3)} = K(4, 3|3)$ .

(2)  $K(14, 9|2) = \overline{K(14, -9|-2)} = K(14, -5|2)$ .

$$(3) K(14, 9| - 2) = \overline{K(14, -9|2)}.$$

$$(4) K(14, -9| - 2) = \overline{K(14, -5|2)}.$$

Note that  $\ell k B(4, -1) = -2$  and  $\ell k B(14, 9) = -1$ . So, we first take the mirror image to make the linking numbers positive, at the expense of changing the sign of  $r$ .

*Proof of Theorem 3.14.* First, we prove the former part, namely we show;

**Proposition 3.16.** *We may assume (1), (2) and (3) of Theorem 3.14.*

*Proof.* Take a 2-bridge link  $B(2\alpha, \beta) = K_1 \cup K_2$ . Without loss of generality, we may assume  $-2\alpha < \beta < 2\alpha$ . If  $\ell k(K_1, K_2) < 0$ , take  $B(2\alpha, -\beta)$ . This corresponds to taking the mirror image of  $B(2\alpha, \beta)$  while preserving the orientation of the components. Therefore,  $\ell k B(2\alpha, -\beta) > 0$ , and hence, from now on, we assume 2-bridge links always have a non-negative linking number. Note that we still have  $-2\alpha < -\beta < 2\alpha$ . Take  $K(2\alpha, \beta|r)$ . Suppose  $r < 0$ . Let  $B(2\alpha', \beta') = K'_1 \cup K'_2$  be the link obtained by taking the mirror image of  $B(2\alpha, \beta)$  while reversing the orientation of  $K_2$ . Now the linking number is preserved, i.e.,  $\ell k B(2\alpha, \beta) = \ell k B(2\alpha', \beta')$ . Recall that  $K(2\alpha, \beta|r)$  is obtained by twisting  $K_1$  by  $K_2$ ,  $r$  times. This does not depend on the orientation of  $K_2$ , and hence the knot obtained by twisting  $K_1$  along  $K_2$   $r$  times is the mirror image of the knot obtained by twisting  $K'_1$  along  $K'_2$   $-r$  times. Therefore, we see  $K(2\alpha, \beta|r) = \overline{K(2\alpha', \beta'|-r)}$   $\square$

Next, we show the following to prove the latter half of Theorem 3.14.

**Proposition 3.17.** *Let  $L = B(2\alpha, \beta)$  be a 2-bridge link, where  $-2\alpha < \beta < 2\alpha$ . Let  $L'$  be obtained by taking the mirror image of  $L$  while reversing the orientation of one component. Then, we have:  $L' = \begin{cases} B(2\alpha, 2\alpha - \beta) & \text{if } \beta > 0, \\ B(2\alpha, -2\alpha - \beta) & \text{if } \beta < 0. \end{cases}$*

*Proof.* Since the other case is similar, we only deal with the case  $\beta > 0$ . Consider the Schubert normal form of  $B(2\alpha, \beta)$ . See Figure 3.3.

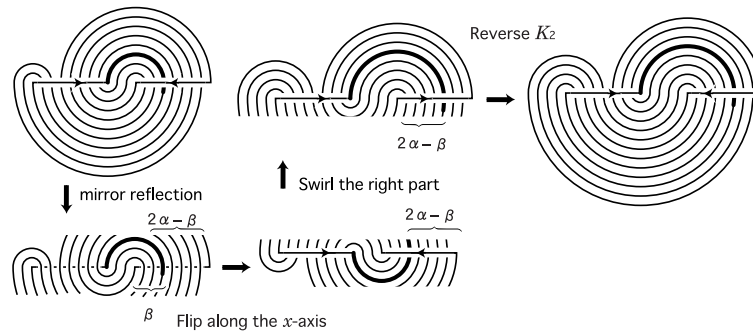


Figure 3.3: Deformation from  $B(2\alpha, \beta)$  to  $B(2\alpha, 2\alpha - \beta)$ , e.g.  $B(8, 3)$  to  $B(8, 5)$

First, take the mirror image by changing all crossings simultaneously. Flip the figure by the horizontal axis. Now we have the Schubert normal form of  $B(2\alpha, -\beta)$ .

Rotate the right over-bridge clockwise by  $\pi$ . Change the orientation of the component containing the right over-bridge. This gives the Schubert normal form of  $B(2\alpha, 2\alpha - \beta)$ . □

By two propositions above, we have Theorem 3.14. □

### 3.5. Primitive spanning disk for $K_1$ .

In this subsection, we introduce the notion of *primitive spanning disk* for  $K_1$ , which locally look like Figure 3.5 (b). This surface is the first step to construct a minimal genus Seifert surface for  $K(2\alpha, \beta|r)$ . Let  $D$  be a diagram obtained from the continued fraction  $S$  of  $B(2\alpha, \beta)$  as in Figure 2.1. By a slight modification of  $D$  corresponding to the modification of  $S$  to  $S^*$ , as in Figure 3.4, construct a spanning disk for  $K_1$ , which consists of horizontal disks and vertical bands, whose interiors are mutually disjoint. In Figure 3.4, each box contains an even number of twists (including 0). Note that in Figure 3.4, all disks are showing the same side, though  $K_2$  may penetrate them from various sides. The set of horizontal disks is divided into several families so that each member of a family meets  $K_2$  from the same side as its neighbouring member(s). This corresponds to the canonical decomposition  $\{P_1, d_1, Q_1, e_1, \dots\}$  of  $S$ . For simplicity, the disks belonging to the family corresponding to  $P_i$ 's (resp.  $Q_i$ 's) are called *positive disks* (resp. *negative disks*), and a band connecting two positive (resp. negative) disks is called a *positive* (resp. *negative*) band. The other bands are called *connecting bands*.

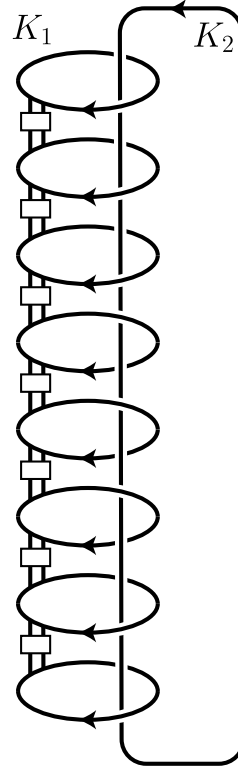


Figure 3.4:  $B(2\alpha, \beta)$

**Remark 3.18.** Disks and bands in the spanning disk for  $K_1$  correspond to edges and vertices, respectively, of the graph  $G(S)$  of  $S$  as follows, except for the end vertices of  $G(S)$ .

- (1) Positive/negative disks correspond to edges with positive/negative slope.
- (2) Positive/negative bands correspond to vertices between positive/negative edges.
- (3) Connecting bands correspond to local maximal or minimal vertices
- (4) The number of twists of a band corresponds to the weight of a vertex.

**Definition 3.19.** Slide each band so that both of its ends are attached to the front edge of each small disk as in Figure 3.5 (b). The *primitive spanning disk* for  $K_1$  is the union of all the small disks together with all bands arranged this way. See Figures 7.2 (a) and 7.4 left.

We remark that in the process of sliding a band, another band may stand in the way. However, as shown in the following proposition, we can always arrange the bands so that each of them appears as in Figure 3.5 (b).

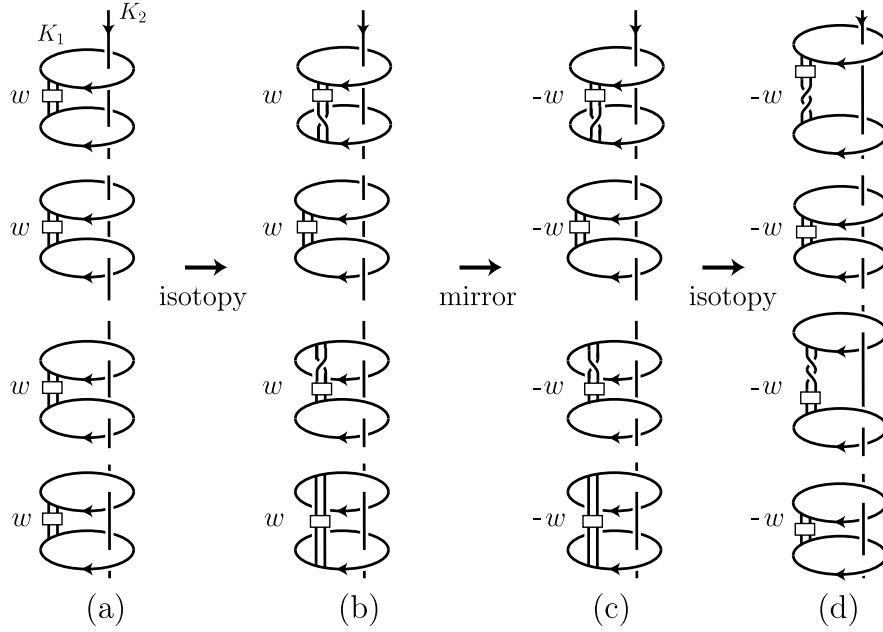


Figure 3.5: Deformation from  $B(2\alpha, \beta)$  to  $B(2\alpha, 2\alpha - \beta)$ .

**Proposition 3.20.** *A relative position of the bands in a primitive spanning disk can be arbitrary.*

*Proof.* Examining the case locally suffices. See Figure 3.6, where disks, say  $D_1, D_2, D_3$  and bands  $B_1, B_2, B_3$  are depicted. To change from (a) to (b), fix  $D_2$  and everything lying above  $D_2$ , and simultaneously turn around everything that hangs below  $D_2$ . Similarly we can change (b) to (c).  $\square$

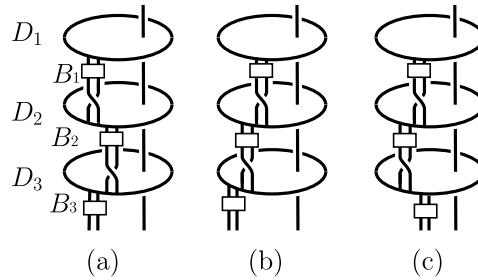


Figure 3.6: Sliding bands to change relative positions

Now, we demonstrate the process of replacing  $B(2\alpha, \beta) = K_1 \cup K_2$  by  $B(2\alpha, 2\alpha - \beta)$ , that is to take the mirror image and reverse (the orientation of)  $K_2$ : Since  $K_2$  consecutively penetrates the disks transversely, we have a diagram of  $K_1$  as in Figure

3.7, where (i) all the disks are concentric, (ii) the higher disk appears smaller and (iii) the only crossings are in the twists of bands. Figure 3.7 shows the process of taking the mirror image and reversing the mirror image of  $K_2$ .

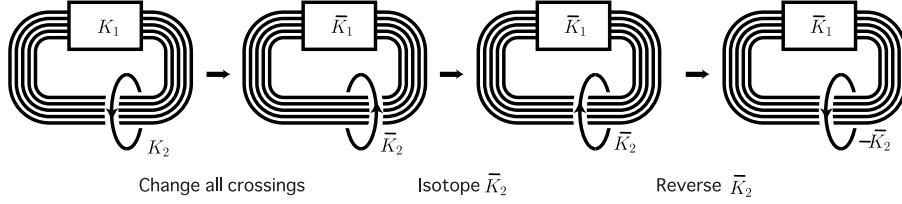


Figure 3.7: Reflect  $K_1 \cup K_2$  and reverse  $K_2$

Then we notice that the effect of the process is simply replacing each of the bands in  $K_1$  by its mirror image. Therefore, the process can be depicted as in Figure 3.5 (b) to (c). Now reversing the operation of (a) to (b), we obtain the standard diagram of the 2-bridge link  $B(2\alpha, 2\alpha - \beta)$ .

Finally, Theorem 3.5 is now almost immediate.

*Proof of Theorem 3.5.* It is easy to see that the final diagram Figure 3.5(d) is the primitive disk obtained from the dual  $\tilde{S}$  of  $S$ . Therefore Theorem 3.5 follows from Proposition 3.17.  $\square$

#### 4. Alexander polynomials (I)

In this section, we determine the Alexander polynomial  $\Delta_{K(r)}(t)$  for  $K(2\alpha, \beta|r)$ . In fact, we prove the following

**Proposition 4.1.** *Let  $\Delta_{B(2\alpha, \beta)}(x, y)$  be the Alexander polynomial of an (oriented) 2-bridge link  $B(2\alpha, \beta)$ . Let  $\Delta_{K(r)}(t)$  be the Alexander polynomial of  $K(2\alpha, \beta|r)$ ,  $r > 0$ .*

- (1) [14] *If  $\ell k B(2\alpha, \beta) = \ell \neq 0$ , then  $\Delta_{K(r)}(t) = \frac{1-t}{1-t^\ell} \Delta_{B(2\alpha, \beta)}(t, t^{\ell r})$*   
(2) *If  $\ell = 0$ , then, for some  $a = \pm 1$  and  $b$ ,*

$$\Delta_{K(r)}(t) = r \left[ \frac{\Delta_{B(2\alpha, \beta)}(x, y)}{1-y} \right]_{x=t, y=1} (1-t) + at^b$$

In Subsection 6.2, we will give a geometric interpretation of  $\Delta_{K(r)}(t)$  when  $\ell = 0$ . (See also [9] or [16])

Now Proposition 4.1 (1) follows from a general result due to Kidwell [14], and hence we omit the proof. However, part(2) was not proved in [14]. In this section, we prove the following more general result suggested by M. Kidwell.

**Proposition 4.2.** *Let  $K_1$  be an oriented knot embedded in an (unknotted) solid torus  $V$ . Suppose  $\ell k(K_1, K_2) = 0$ , where  $K_2$  is an oriented meridian of  $\partial V$ . Denote by  $K_1(r)$  the knot obtained from  $K_1$  by applying Dehn twists  $r$  times along  $K_2$*

( $r > 0$ ). Let  $L = K_1 \cup K_2$ . Then, for some  $a = \pm 1$  and  $b$ , we have:

$$\Delta_{K_1(r)}(t) = r(1-t) \left[ \frac{\Delta_L(x,y)}{1-y} \right]_{\substack{x=t \\ y=1}} + at^b \Delta_{K_1}(t). \quad (4.1)$$

Proposition 4.1 (2) follows from Proposition 4.2 immediately, since  $\Delta_{K_1}(t) = 1$ .

*Proof of Proposition 4.2.* First, consider the link  $L = K_1 \cup K_2$ . We add one trivial knot  $K_3$  to  $L$  such that  $\ell k(K_1, K_3) = \ell k(K_2, K_3) = 1$  as in Figure 4.1. Let  $\tilde{L} = K_1 \cup K_2 \cup K_3$  be the 3-component link.

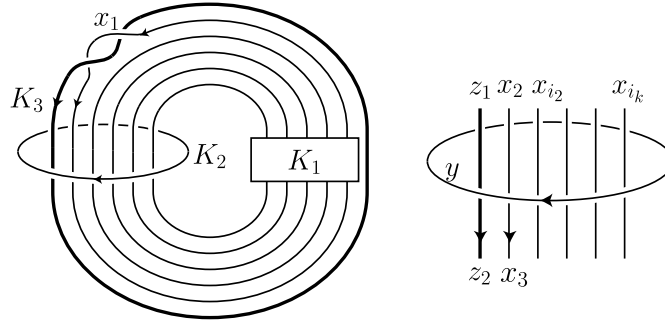


Figure 4.1: A diagram of  $\tilde{L} = K_1 \cup K_2 \cup K_3$

Using this diagram, we obtain the following Wirtinger presentation of the link group  $G(\tilde{L})$  of  $\tilde{L}$ .

$$G(\tilde{L}) = \langle x_1, x_2, \dots, x_m, y, z_1, z_2 | r_1, \dots, r_m, s, t_1, t_2 \rangle, \text{ where}$$

$$\begin{aligned} r_1 &= z_1 x_1 z_1^{-1} x_2^{-1}, & s &= (x_{i_k}^{\varepsilon_k} \cdots x_{i_2}^{\varepsilon_2} x_2 z_1) y (z_1^{-1} x_2^{-1} x_{i_2}^{-\varepsilon_2} \cdots x_{i_k}^{-\varepsilon_k}) y^{-1}, \\ r_2 &= y x_2 y^{-1} x_3^{-1}, & t_1 &= y z_1 y^{-1} z_2^{-1}, \\ r_3 &= w_3 x_3 w_3^{-1} x_4^{-1}, & t_2 &= x_1 z_2 x_1^{-1} z_1^{-1} \\ &\vdots & & \\ r_m &= w_m x_m w_m^{-1} x_1^{-1}. \end{aligned}$$

Here,  $w_i$  is a word in  $x_i$  and/or  $y$ .

We note that  $\varepsilon_k + \cdots + \varepsilon_2 + 1 = 0$ , since  $\ell k(K_1, K_2) = 0$ .

Now using  $t_1$  and  $t_2$ , we can eliminate  $z_2$  and obtain a new presentation. For simplicity, we write  $z = z_1$ . Then

$$G(\tilde{L}) = \langle x_1, x_2, \dots, x_m, y, z | r_1, \dots, r_m, s, t' \rangle, \text{ where } t' = x_1 y z y^{-1} x_1^{-1} z^{-1}.$$

From this presentation, we obtain the Alexander matrix  $M(\tilde{L})$  for  $\tilde{L}$ . The matrix  $M(\tilde{L})$  is an  $(m+2) \times (m+2)$  matrix. A simple calculation shows

$$\begin{aligned} (1) \quad & \left( \frac{\partial r_i}{\partial y} \right)^\phi = \delta_i (1-x), \text{ where } \delta_i = 0, 1 \text{ or } -y^{-1}, \\ (2) \quad & \left( \frac{\partial r_1}{\partial z} \right)^\phi = 1-x, \left( \frac{\partial r_i}{\partial z} \right)^\phi = 0, \text{ for } i \neq 1, \end{aligned} \quad (4.2)$$

where  $\partial$  indicates Fox's free derivative and  $\phi$  is the induced homomorphism from  $G(\tilde{L})$  to the free abelian group  $G(\tilde{L})/[G(\tilde{L}), G(\tilde{L})]$ , where  $x_i^\phi = x, y^\phi = y$  and  $z^\phi = z$ . Let  $U = x_{i_k}^{\varepsilon_k} \cdots x_{i_2}^{\varepsilon_2} x_2 z$ . Then  $s = UyU^{-1}y^{-1}$ , and

$$\left(\frac{\partial s}{\partial x_i}\right)^\phi = (1-y)\left(\frac{\partial U}{\partial x_i}\right)^\phi. \quad (4.3)$$

Since  $U$  does not involve  $y$  and  $\varepsilon_k + \cdots + \varepsilon_2 + 1 = 0$ , we see

$$\begin{aligned} (1) \quad & \left(\frac{\partial s}{\partial y}\right)^\phi = z - 1 \\ (2) \quad & \left(\frac{\partial s}{\partial z}\right)^\phi = 1 - y \end{aligned} \quad (4.4)$$

Furthermore, we have:

$$\begin{aligned} (1) \quad & \left(\frac{\partial t'}{\partial x_1}\right)^\phi = 1 - z, \quad \left(\frac{\partial t'}{\partial x_i}\right)^\phi = 0, \text{ for } i \neq 1, \\ (2) \quad & \left(\frac{\partial t'}{\partial y}\right)^\phi = x(1 - z), \\ (3) \quad & \left(\frac{\partial t'}{\partial z}\right)^\phi = xy - 1. \end{aligned} \quad (4.5)$$

Now, the Alexander polynomial  $\Delta_{\tilde{L}}(x, y, z)$  of  $\tilde{L}$  is obtained as follows.

Denote by  $\widehat{M}(\tilde{L})$  the  $(m+1) \times (m+2)$  matrix obtained from  $M(\tilde{L})$  by striking out the  $m^{\text{th}}$  row:  $\left(\left(\frac{\partial r_m}{\partial x_1}\right)^\phi, \dots, \left(\frac{\partial r_m}{\partial x_m}\right)^\phi, \left(\frac{\partial r_m}{\partial y}\right)^\phi, \left(\frac{\partial r_m}{\partial z}\right)^\phi\right)$

Further,  $\widehat{M}(\tilde{L})_\nu$  denotes the  $(m+1) \times (m+1)$  matrix obtained from  $\widehat{M}(\tilde{L})$  by striking out the column corresponding to the generator  $\nu$ . (For instance, to get  $\widehat{M}(\tilde{L})_z$ , eliminate the last column of  $\widehat{M}(\tilde{L})$ .) Then the following is known:

$$\Delta_{\tilde{L}}(x, y, z) \doteq \frac{\det \widehat{M}(\tilde{L})_z}{1 - z}. \quad (4.6)$$

Since the last row of  $\widehat{M}(\tilde{L})_z$  is divisible by  $1 - z$ , we have:

$$\Delta_{\tilde{L}}(x, y, z) = \det \left[ \begin{array}{c|c} \left(\frac{\partial r_i}{\partial x_j}\right)^\phi & \delta_i(1-x) \\ \hline (1-y)\left(\frac{\partial U}{\partial x_j}\right)^\phi & z-1 \\ \hline 1 & 0 \cdots 0 & x \end{array} \right]. \quad (4.7)$$

Let  $\widehat{L}(r)$  be the link obtained from  $K_1 \cup K_3$  by applying Dehn twists  $r(> 0)$  times along  $K_2$ .



Since  $\ell k(K_2, K_1) = 0$  and  $\ell k(K_2, K_3) = 1$ , by Kidwell's theorem [14, Corollary 3.2] we have:

$$\Delta_{\widehat{L}}(x, z) = \frac{1}{1-z} \det \left[ \begin{array}{c|c} \left( \frac{\partial r_i}{\partial x_j} \right)_{y=z^r}^\phi & \delta_i(1-x) \\ \hline (1-z^r) \left( \frac{\partial U}{\partial x_j} \right)^\phi & z-1 \\ \hline 1 \ 0 \ 0 \ \cdots \ 0 & x \end{array} \right] \quad (4.8)$$

Further, our knot  $K_1(r)$  is obtained from  $\widehat{L}$  by eliminating  $K_3$ , and hence, by Torres' Theorem [19], noting  $\ell k(K_3, K_1(r)) = 1$ , we have:

$$\Delta_{K_1(r)}(x) = \Delta_{\widehat{L}}(x, 1), \text{ and hence,} \quad (4.9)$$

$$\Delta_{K_1(r)}(x) = \det \left[ \begin{array}{c|c} \left( \frac{\partial r_i}{\partial x_j} \right)^\phi & \delta_i(1-x) \\ \hline r \left( \frac{\partial U}{\partial x_j} \right)^\phi & -1 \\ \hline 1 \ 0 \ 0 \ \cdots \ 0 & x \end{array} \right]_{y=z=1} \quad (4.10)$$

We evaluate  $\Delta_{K_1(r)}(x)$  by expanding it along the last row, and hence

$$\Delta_{K_1(r)}(x) \doteq \det \left[ \begin{array}{c|c} \left( \frac{\partial r_i}{\partial x_j} \right)_{\substack{i \geq 1 \\ j \geq 2}}^\phi & \delta_i(1-x) \\ \hline r \left( \frac{\partial U}{\partial x_j} \right)_{j \geq 2}^\phi & -1 \end{array} \right]_{y=z=1} + (-1)^m x \det \left[ \begin{array}{c|c} \left( \frac{\partial r_i}{\partial x_j} \right)_{\substack{i \geq 1 \\ j \geq 1}}^\phi & \\ \hline r \left( \frac{\partial U}{\partial x_j} \right)_{j \geq 1}^\phi & \end{array} \right]_{y=z=1} \quad (4.11)$$

First we claim:

**Lemma 4.3.**  $\det \left[ \begin{array}{c|c} \left( \frac{\partial r_i}{\partial x_j} \right)_{\substack{i \geq 1 \\ j \geq 1}}^\phi & \\ \hline r \left( \frac{\partial U}{\partial x_j} \right)_{j \geq 1}^\phi & \end{array} \right]_{y=z=1} = 0$

*Proof.* Since  $y = 1$  and  $\varepsilon_k + \cdots + \varepsilon_2 + 1 = 0$ , we have  $\sum_{j=1}^m \left( \frac{\partial r_i}{\partial x_j} \right)_{y=1}^\phi = 0$  and  $\sum_{j=1}^m \left( \frac{\partial U}{\partial x_j} \right)_{y=1}^\phi = 0$ , and hence Lemma 4.3 follows.  $\square$

Now we return to the proof of Proposition 4.2. From (4.11) and Lemma 4.3, we see the following:

$$\Delta_{K_1(r)}(x) \doteq \det \left[ \begin{array}{c|c} \left( \frac{\partial r_i}{\partial x_j} \right)_{\substack{i \geq 1 \\ j \geq 2}}^\phi & \delta_i(1-x) \\ \hline r \left( \frac{\partial U}{\partial x_j} \right)_{j \geq 2}^\phi & -1 \end{array} \right]_{y=1} \quad (4.12)$$

The determinant is decomposed into two terms as follows:

$$\Delta_{K_1(r)}(x) \doteq \det \left[ \frac{\left( \frac{\partial r_i}{\partial x_j} \right)_{\substack{i \geq 1 \\ j \geq 2}}^\phi \left| \delta_i(1-x) \right.}{r \left( \frac{\partial U}{\partial x_j} \right)_{j \geq 2}^\phi \left| 0 \right.} \right]_{y=1} + \det \left[ \frac{\left( \frac{\partial r_i}{\partial x_j} \right)_{\substack{i \geq 1 \\ j \geq 2}}^\phi \left| 0 \right.}{r \left( \frac{\partial U}{\partial x_j} \right)_{j \geq 2}^\phi \left| -1 \right.} \right]_{y=1} \quad (4.13)$$

The second term is equivalent to

$\det \left[ \left( \frac{\partial r_i}{\partial x_j} \right)_{\substack{1 \leq i \leq m-1 \\ 2 \leq j \leq m}}^\phi \right]_{y=1}$  that is equal to  $\Delta_{K_1}(x)$  (up to  $\pm x^k$ ). Therefore, the final step is to show that

$$\det \left[ \frac{\left( \frac{\partial r_i}{\partial x_j} \right)_{\substack{i \geq 1 \\ j \geq 2}}^\phi \left| \delta_i \right.}{\left( \frac{\partial U}{\partial x_j} \right)_{j \geq 2}^\phi \left| 0 \right.} \right]_{y=1} \doteq \left[ \frac{\Delta_{B(2\alpha, \beta)}(x, y)}{1-y} \right]_{y=1}. \quad (4.14)$$

To show (4.14) we go back to  $M(\tilde{L})$  and compute  $\Delta_{\tilde{L}}(x, y, z)$  in a different way. We use the following formula:

$$\Delta_{\tilde{L}}(x, y, z) = \frac{\det \widehat{M}(\tilde{L})_y}{1-y} \quad (4.15)$$

Then the row  $(\frac{\partial s}{\partial x_1}, \frac{\partial s}{\partial x_2}, \dots, \frac{\partial s}{\partial x_m}, \frac{\partial s}{\partial z})^\phi$  is divisible by  $1-y$ , and hence, we have:

$$\Delta_{\tilde{L}}(x, y, z) = \det \left[ \begin{array}{c|c} \left( \frac{\partial r_i}{\partial x_j} \right)_{j \geq 1}^\phi & \begin{array}{c} 1-x \\ 0 \\ \vdots \\ 0 \end{array} \\ \hline \left( \frac{\partial U}{\partial x_j} \right)_{j \geq 1}^\phi & 1 \\ \hline 1-z & 0 \cdots 0 \mid xy-1 \end{array} \right] \quad (4.16)$$

Now we try to find  $\Delta_L(x, y)$  from  $\Delta_{\tilde{L}}(x, y, z)$ .

To do this, we eliminate  $K_3$  from  $\tilde{L} = K_1 \cup K_2 \cup K_3$ . Then, since  $\ell k(K_3, K_1) = \ell k(K_3, K_2) = 1$ , Torres' Theorem [19] implies:

$$\Delta_L(x, y) = \frac{\Delta_{\tilde{L}}(x, y, 1)}{xy-1}, \quad (4.17)$$

that is, from (4.16),

$$\Delta_L(x, y) = \det \left[ \begin{array}{c} \left( \frac{\partial r_i}{\partial x_j} \right)_{j \geq 1}^\phi \\ \left( \frac{\partial U}{\partial x_j} \right)_{j \geq 1}^\phi \end{array} \right]_{z=1} = N \quad (4.18)$$

We describe  $N$  precisely. First we note:

$$\begin{aligned} (1) \sum_{j=1}^m \left( \frac{\partial r_i}{\partial x_j} \right)^\phi &= \begin{cases} y^\varepsilon - 1, & \text{if } r_i \text{ is of the form } : y^\varepsilon x_i y^{-\varepsilon} x_{i+1}^{-1}, \varepsilon = \pm 1 \\ 0, & \text{otherwise.} \end{cases} \\ (2) \sum_{j=1}^m \left( \frac{\partial U}{\partial x_j} \right)^\phi &= 0. \end{aligned} \quad (4.19)$$

Therefore, if we add all columns of  $N$  to the first column to get  $N_1$ , then the first column of  $N_1$  is divisible by  $1 - y$ . Further,

$$\sum_{j=1}^m \left( \frac{\partial r_i}{\partial x_j} \right)^\phi = \varepsilon y^{\frac{\varepsilon-1}{2}} (1 - y). \quad (4.20)$$

Since  $\varepsilon y^{\frac{\varepsilon-1}{2}} = \delta_i$ , we have:

$$\frac{N_1}{1 - y} = (-1)^m \det \left[ \frac{\left( \frac{\partial r_i}{\partial x_j} \right)^\phi_{j \geq 2} \middle| \delta_i}{\left( \frac{\partial U}{\partial x_j} \right)^\phi_{j \geq 2} \middle| 0} \right] = \frac{\Delta_L(x, y)}{1 - y}. \quad (4.21)$$

Evaluations of both polynomials at  $y = 1$  give (4.14).

The proof of Proposition 4.2 is now completed.  $\square$

## 5. Alexander polynomials (II)

We have established some relationships between the Alexander polynomial of  $K(2\alpha, \beta|r)$  and that of the 2-bridge link  $B(2\alpha, \beta)$ . However, these relations are not sufficient to our purpose. Therefore, in this section, we prove some subtle properties of  $\Delta_{B(2\alpha, \beta)}(x, y)$ . These properties are indispensable to study the Alexander polynomial of our knot  $K(2\alpha, \beta|r)$ . See Theorem 5.5.

Let  $S = \{P_1, d_1, Q_1, e_1, P_2, \dots, P_m, d_m, Q_m\}$  be the canonical decomposition of the continued fraction of  $\beta/2\alpha$ . Let  $\rho_i, \rho_j, \rho, \lambda_i, \lambda_j$  and  $\lambda$  be integers as defined in Definition 3.10. Now, by Proposition 3.11, we can write

$$\Delta_{B(2\alpha, \beta)}(x, y) = f_{\lambda-1}(x)y^{\lambda-1} + f_{\lambda-2}(x)y^{\lambda-2} + \dots + f_0(x), \quad (5.1)$$

where  $f_i(x), 0 \leq i \leq \lambda - 1$ , are integer polynomials in  $x$  of degree at most  $\lambda - 1$ .

Our purpose is to determine these polynomials  $f_i(x)$ , in particular,  $f_{\lambda-1}(x)$ .

### 5.1 Skein relation

Let  $[[u_1, v_1, u_2, v_2, \dots, u_s, v_s, u_{s+1}]]$  be the continued fraction of  $\beta/2\alpha$ . Then it is shown in [12, Theorem 2 (4.2)] that

$$\begin{aligned} &\Delta_{B(2\alpha, \beta)}(x, y) \\ &= v_s(x-1)(y-1)F_{u_{s+1}}(x, y)\Delta[[u_1, v_1, \dots, u_s]] - \Delta[[u_1, v_1, \dots, v_{s-1}, u_s + u_{s+1}]], \end{aligned} \quad (5.2)$$

where  $\Delta[[c_1, \dots, c_k]]$  is the Alexander polynomial of the 2-bridge link associated to the continued fraction  $[[c_1, c_2, \dots, c_k]]$ , and  $F_n(x, y)$  is defined below:

$$\begin{aligned}
(1) \quad & F_0(x, y) = 0. \\
(2) \quad & \text{For } n > 0, \\
& (a) \quad F_n(x, y) = 1 + xy + \dots + (xy)^{n-1} = \frac{(xy)^n - 1}{xy - 1}, \\
& (b) \quad F_{-n}(x, y) = -\{(xy)^{-1} + \dots + (xy)^{-n}\} = \frac{-1}{(xy)^n} F_n(x, y). \quad (5.3)
\end{aligned}$$

Note that  $F_c(x, y) = \Delta[[c]]$ .

Formula (5.2) is obtained by applying crossing changes and smoothing at  $v_s$ , i.e., at the crossings corresponding to  $v_s$ .

We should note that (5.2) is slightly different from the original formula given in [12, (4.2)], since we use a different notation.

By applying (5.2) on all  $v_j, j = 1, 2, \dots, s$ , we obtain  $\Delta_{B(2\alpha, \beta)}(x, y)$  in terms of various  $\Delta[[c]]$ , where  $c$  is written as the sum of  $u_i$ .

The following example illustrates a calculation.

**Example 5.1.** Write  $\frac{\beta}{2\alpha} = [[u_1, v_1, u_2, v_2, u_3]]$ . Then,

$$\begin{aligned}
& \Delta_{B(2\alpha, \beta)} \\
&= v_2(x-1)(y-1)F_{u_3}(x, y)\Delta[[u_1, v_1, u_2]] - \Delta[[u_1, v_1, u_2 + u_3]] \\
&= v_2(x-1)(y-1)F_{u_3}(x, y)\{v_1(x-1)(y-1)F_{u_2}(x, y)F_{u_1}(x, y) - \Delta[[u_1 + u_2]]\} \\
&\quad - \{v_1(x-1)(y-1)F_{u_2+u_3}(x, y)\Delta[[u_1]] - \Delta[[u_1 + u_2 + u_3]]\} \\
&= v_1v_2(x-1)^2(y-1)^2F_{u_1}F_{u_2}F_{u_3} - (x-1)(y-1)\{v_1F_{u_1}F_{u_2+u_3} + v_2F_{u_1+u_2}F_{u_3}\} \\
&\quad + F_{u_1+u_2+u_3}
\end{aligned}$$

As is illustrated in Example 5.1, we see that  $\Delta_{B(2\alpha, \beta)}(x, y)$  is of the following form:

$$\Delta_{B(2\alpha, \beta)}(x, y) = \sum_{\substack{0 \leq k \leq s \\ 1 \leq i_1 < i_2 < \dots < i_k \leq s}} (-1)^k v_{i_1} v_{i_2} \dots v_{i_k} (x-1)^k (y-1)^k F_{\mu_1} F_{\mu_2} \dots F_{\mu_k}, \quad (5.4)$$

where the summation is taken over all indices  $i_j$  such that  $1 \leq i_1 < i_2 < \dots < i_k \leq s$ , and  $\mu_j$  is of the form:  $\mu_j = u_{j_1} + u_{j_1+1} + \dots + u_{j_1+p}$  and  $\mu_1 + \mu_2 + \dots + \mu_{k+1} = u_1 + u_2 + \dots + u_{s+1}$ .

For convenience, we denote by  $\Lambda_{p,r}$  the set of all  $p$  indices  $i_1, \dots, i_p$  such that  $1 \leq i_1 < \dots < i_p \leq r$ . Since  $F_c(x, y)$  is a rational function, we replace  $F_c(x, y)$  by a polynomial  $\tilde{F}_c(x, y)$  below.

For  $n > 0$ ,

$$\begin{aligned}
(1) \quad & \tilde{F}_n(x, y) = (xy-1)F_n(x, y) = (xy)^n - 1. \\
(2) \quad & \tilde{F}_{-n}(x, y) = (xy)^n(xy-1)F_{-n}(x, y) = (-1)\tilde{F}_n(x, y) = (-1)\{(xy)^n - 1\}. \quad (5.5)
\end{aligned}$$

Using these polynomials, we obtain an integer polynomial  $\tilde{\Delta}_{B(2\alpha,\beta)}(x, y)$  from  $\Delta_{B(2\alpha,\beta)}(x, y)$ :

$$\tilde{\Delta}_{B(2\alpha,\beta)}(x, y) = (xy)^{\sum_{j=1}^m \lambda'_j} (xy - 1)^{\sum_i (s_i + 1) + \sum_j (q_j + 1)} \Delta_{B(2\alpha,\beta)}(x, y). \quad (5.6)$$

Therefore we have:

$$\begin{aligned} (1) \quad & \max y\text{-deg } \tilde{\Delta}_{B(2\alpha,\beta)}(x, y) \\ &= \max y\text{-deg } \Delta_{B(2\alpha,\beta)}(x, y) + \sum_{i=1}^m s_i + \sum_{j=1}^m q_j + 2m \\ &= \lambda + \sum_{i=1}^m s_i + \sum_{j=1}^m q_j + 2m - 1, \\ (2) \quad & \min y\text{-deg } \tilde{\Delta}_{B(2\alpha,\beta)}(x, y) = 0. \end{aligned} \quad (5.7)$$

Now we can write

$$\begin{aligned} \tilde{\Delta}_{B(2\alpha,\beta)}(x, y) &= \tilde{f}_\nu(x) y^\nu + \cdots + \tilde{f}_0(x), \text{ and} \\ \tilde{f}_\nu(x) &= f_{\lambda-1}(x) x^{\sum_{i=1}^m s_i + \sum_{j=1}^m q_j + 2m}, \end{aligned} \quad (5.8)$$

where  $\nu = \lambda + \sum_{i=1}^m s_i + \sum_{j=1}^m q_j + 2m - 1$ .

First we show

$$\deg \tilde{f}_\nu(x) = \lambda + \sum_{i=1}^m s_i + \sum_{j=1}^m q_j - \rho + 2m - 1. \quad (5.9)$$

## 5.2 Proof of (5.9) (I)

We consider two special cases.

Case 1. All  $u_i > 0$ .

Consider  $\beta/2\alpha = [[u_1, v_1, u_2, v_2, \dots, u_s, v_s, u_{s+1}]]$ .

Then  $\Delta_{B(2\alpha,\beta)}(x, y) = \sum_{\Lambda_{k,s}, 0 \leq k \leq s} (-1)^k v_{i_1} v_{i_2} \cdots v_{i_k} (x-1)^k (y-1)^k F_{\mu_1} F_{\mu_2} \cdots F_{\mu_{k+1}},$

where  $\mu_i > 0, 1 \leq i \leq k+1$ , and  $\lambda = \sum_{i=1}^{k+1} \mu_i, \lambda' = 0$ . Therefore,

$$\begin{aligned} & \tilde{\Delta}_{B(2\alpha,\beta)}(x, y) \\ &= (xy - 1)^{s+1} \Delta_{B(2\alpha,\beta)}(x, y) \\ &= \sum_{\substack{\Lambda_{k,s} \\ 0 \leq k \leq s}} (-1)^k v_{i_1} v_{i_2} \cdots v_{i_k} (x-1)^k (y-1)^k \tilde{F}_{\mu_1} \tilde{F}_{\mu_2} \cdots \tilde{F}_{\mu_{k+1}} (xy - 1)^{s-k}. \end{aligned} \quad (5.10)$$

Case 2. All  $-u_i < 0$ .

Consider  $\beta/2\alpha = [[-u_1, -v_1, -u_2, -v_2, \dots, -u_q, -v_q, -u_{q+1}]]$ . Then  $\Delta_{B(2\alpha,\beta)}(x, y) = (-1)^k (-v_{i_1}) (-v_{i_2}) \cdots (-v_{i_k}) (x-1)^k (y-1)^k F_{-\mu_1} F_{-\mu_2} \cdots F_{-\mu_{k+1}},$

and hence  $\lambda' = u_1 + u_2 + \cdots + u_{q+1} = \mu_1 + \mu_2 + \cdots + \mu_{k+1}$ . Therefore:

$$\begin{aligned}
& \tilde{\Delta}_{B(2\alpha, \beta)}(x, y) \\
&= (xy)^{\lambda'} (xy - 1)^{q+1} \Delta_{B(2\alpha, \beta)}(x, y) \\
&= \sum_{\Lambda_{k,q}, 0 \leq k \leq q} v_{i_1} v_{i_2} \cdots v_{i_k} (-1)^{k+1} (x-1)^k (y-1)^k \tilde{F}_{\mu_1} \tilde{F}_{\mu_2} \cdots \tilde{F}_{\mu_{k+1}} (xy-1)^{q-k} \\
&= - \sum_{\Lambda_{k,q}, 0 \leq k \leq q} (-1)^k v_{i_1} v_{i_2} \cdots v_{i_k} (x-1)^k (y-1)^k \tilde{F}_{\mu_1} \tilde{F}_{\mu_2} \cdots \tilde{F}_{\mu_{k+1}} (xy-1)^{q-k}.
\end{aligned} \tag{5.11}$$

Note that (5.10) and (5.11) are of the same form.

Now consider the general case. Let  $\{P_1, d_1, Q_1, e_1, P_2, \dots, P_m, d_m, Q_m\}$  be the canonical decomposition of the continued fraction of  $\beta/2\alpha$ .

Denote  $P_i = [[a_{i,1}, b_{i,1}, a_{i,2}, b_{i,2}, \dots, a_{i,s_i}, b_{i,s_i}, a_{i,s_i+1}]]$ ,  $1 \leq i \leq m$ , and  $Q_j = [[-a'_{j,1}, -b'_{j,1}, -a'_{j,2}, -b'_{j,2}, \dots, -a'_{j,q_j}, -b'_{j,q_j}, -a'_{j,q_j+1}]]$ ,  $1 \leq j \leq m$ , where  $a_{i,p} > 0$  and  $a'_{j,p} > 0$ , but  $b_{i,q}, b'_{j,q}$  are arbitrary. Then by Proposition 3.11,

$$\max y\text{-deg } \Delta_{B(2\alpha, \beta)}(x, y) = \sum_{i=1}^m \sum_{k=1}^{s_i+1} a_{i,k} + \sum_{j=1}^m \sum_{k=1}^{q_j+1} a'_{j,k} - 1 = \lambda - 1.$$

First we try to find the term with the max  $y$ -degree in  $\tilde{\Delta}_{B(2\alpha, \beta)}(x, y)$ .

Denote by  $\Delta_{P_i}(x, y)$  (resp.  $\Delta_{Q_j}(x, y)$ ) the Alexander polynomial of the 2-bridge link associated to  $P_i$  (resp.  $Q_j$ ). Then, as we did above, we obtain

$$\Delta_{P_i}(x, y) = \sum_{\Lambda_{k,s_i}, 0 \leq k \leq s_i} (-1)^k b_{i,p_1} b_{i,p_2} \cdots b_{i,p_k} (x-1)^k (y-1)^k F_{\mu_1} F_{\mu_2} \cdots F_{\mu_{k+1}},$$

where  $\mu_1 + \mu_2 + \cdots + \mu_{k+1} = \lambda_i$ , and hence,

$$\begin{aligned}
& \tilde{\Delta}_{P_i}(x, y) \\
&= (xy-1)^{s_i+1} \Delta_{P_i}(x, y) \\
&= \sum_{\Lambda_{k,s_i}} (-1)^k b_{i,p_1} b_{i,p_2} \cdots b_{i,p_k} (x-1)^k (y-1)^k \tilde{F}_{\mu_1} \tilde{F}_{\mu_2} \cdots \tilde{F}_{\mu_{k+1}} (xy-1)^{s_i-k},
\end{aligned} \tag{5.12}$$

where  $\tilde{F}_{\mu} = (xy)^{\mu} - 1$ ,  $\mu > 0$ .

On the other hand,

$$\Delta_{Q_j}(x, y) = \sum_{\Lambda_{k,q_j}} (-1)^k b'_{j,r_1} b'_{j,r_2} \cdots b'_{j,r_k} (x-1)^k (y-1)^k F_{-\mu'_1} F_{-\mu'_2} \cdots F_{-\mu'_{k+1}}, \text{ and}$$

hence we have:

$$\begin{aligned}
& \tilde{\Delta}_{Q_j}(x, y) \\
&= (xy)^{\lambda'} (xy-1)^{q_j+1} \Delta_{Q_j}(x, y) \\
&= - \sum_{\Lambda_{k,q_j}} (-1)^k b'_{j,r_1} b'_{j,r_2} \cdots b'_{j,r_k} (x-1)^k (y-1)^k \tilde{F}_{\mu'_1} \tilde{F}_{\mu'_2} \cdots \tilde{F}_{\mu'_{k+1}} (xy-1)^{q_j-k}.
\end{aligned} \tag{5.13}$$

### 5.3. Proof of (5.9) (II)

To evaluate  $\Delta_B(x, y)$ , we must split and smooth at various crossings. We classify these operations into two types.

Type 1. Split all crossings at every  $d_i$  and  $e_j$ .

Type 2 Smooth some crossings at some  $d_i$  and/or  $e_j$ .

From Type 1 operation, we obtain the following term in  $\tilde{\Delta}_B(x, y)$  :

$$A = (-1)^m d_1 \cdots d_m (-1)^{m-1} e_1 \cdots e_{m-1} (x-1)^{2m-1} (y-1)^{2m-1} \\ \times \prod_{i=1}^m \tilde{\Delta}_{P_i}(x, y) \prod_{j=1}^m \tilde{\Delta}_{Q_j}(x, y). \quad (5.14)$$

Terms in  $A$  with the max  $y$ -degree are obtained by

- (1) taking  $y^{2m-1}$  from  $(y-1)^{2m-1}$ ,
- (2) taking , in each  $P_i$ ,  $y^k$  from  $(y-1)^k$ ,  $(xy)^{\mu_i}$  from each  $\tilde{F}_{\mu_i}$  and  $(xy)^{s_i-k}$  from  $(xy-1)^{s_i-k}$ , and
- (3) taking, in each  $Q_j$ ,  $y^k$  from  $(y-1)^k$ ,  $(xy)^{\mu'_i}$  from  $\tilde{F}_{\mu'_i}$ , and  $(xy)^{q_j-k}$  from  $(xy-1)^{q_j-k}$ .

Therefore, the max  $y$ -degree in  $A$  is

$$2m-1 + \sum_{i=1}^m (k + \mu_1 + \cdots + \mu_{k+1} + s_i - k) + \sum_{j=1}^m (k + \mu'_1 + \cdots + \mu'_{k+1} + q_j - k) \\ = 2m-1 + \sum_{i=1}^m (s_i + \lambda_i) + \sum_{j=1}^m (q_j + \lambda'_j) = 2m-1 + \sum_{i=1}^m s_i + \sum_{j=1}^m q_j + \lambda.$$

While, the min  $y$ -deg in  $A$  is obviously 0.

Since  $\tilde{\Delta}_B(x, y) = (xy)^{\sum_{i=1}^m \lambda'_i} (xy-1)^{\sum_{i=1}^m (s_i+1) + \sum_{j=1}^m (q_j+1)} \Delta_B(x, y)$ ,

the  $y$ -degree of  $\Delta_B(x, y)$  is at least  $2m-1 + \sum s_i + \sum q_j + \lambda - (\sum s_i + m + \sum q_j + m) = \lambda - 1$ , that coincides with Proposition 3.11. Therefore, these terms are in fact the terms with maximal  $y$ -degree.

### 5.4. Proof of (5.9) (III)

Next we show that Type 2 operation does not yield a term with max  $y$ -degree in  $\tilde{\Delta}_B(x, y)$ . To see this, we can assume without loss of generality that we smooth only crossings at  $d_1$ , but not others. Namely, we split at other crossings  $d_i (i \neq 1)$  and  $e_j, 1 \leq j \leq m-1$ .

Case 1. Suppose  $a_{1,s_1+1} > a'_{1,1}$ .

By smoothing at  $d_1$ , we have a new canonical decomposition of the new continued fraction:  $\hat{S} = \{\hat{P}_1, \hat{c}_1, \hat{Q}_1, e_1, P_2, d_2, Q_2, e_2, \cdots, P_m, d_m, Q_m\}$ , where

$\hat{P}_1 = [[a_{1,1}, b_{1,1}, a_{1,2}, b_{1,2}, \cdots, a_{1,s_1}, b_{1,s_1}, a_{1,s_1+1} - a'_{1,1}]]$ ,  $\hat{c}_1 = -b'_{1,1}$  and

$\hat{Q}_1 = [[-a'_{1,2}, -b'_{1,2}, \cdots, -a'_{1,q_1}, -b'_{1,q_1}, -a'_{1,q_1+1}]]$ .

Consider  $\tilde{\Delta}_B(x, y) = (xy)^{\lambda'} (xy-1)^{\sum (s_i+1) + \sum (q_j+1)} \Delta_B(x, y)$ . Using the previous argument, we can determine the terms of max  $y$ -degree of  $\Delta(\hat{S})$  in  $\tilde{\Delta}_B(x, y)$ . Since the terms of max  $y$ -degree are obtained as those in each  $P_i$  and  $Q_j$ , we will determine

these terms for  $\widehat{P}_1$  and  $\widehat{Q}_1$ . For  $\widehat{P}_1$ , the max  $y$ -degree is

$$k + \widehat{\mu}_1 + \cdots + \widehat{\mu}_{k+1} + s_1 + 1 - (k + 1). \quad (5.15)$$

Since  $\widehat{\mu}_1 + \cdots + \widehat{\mu}_{k+1} = a_{1,1} + a_{1,2} + \cdots + a_{1,s_1+1} - a'_{1,1} = \lambda_1 - a'_{1,1}$ , it follows from (5.15) that the max  $y$ -degree is  $\lambda_1 - a'_{1,1} + s_1$ . For  $\widehat{Q}_1$ , the maximal terms are contained in

$\sum (-1)^k b'_{1,r_1} \cdots b'_{1,r_k} (x-1)^k (y-1)^k \widetilde{F}_{\mu'_1} \cdots \widetilde{F}_{\mu'_{k+1}} (xy-1)^{q_1-k-1} (xy)^{a'_{1,1}} (xy-1)$ , where the sum is taken over  $2 \leq r_1 < \cdots < r_k \leq q_1, k = 0, 1, \dots, q_1 - 1$ .

Since the original multipliers  $(xy)^{\lambda'}$  cannot be cancelled out in this case,  $(xy)^{a'_{1,1}}$  remains. Therefore, max  $y$ -degree in  $\widehat{Q}_1$  is  $k + q_1 - k - 1 + \widehat{\mu}'_1 + \cdots + \widehat{\mu}'_{k+1} + a'_{1,1} + 1 = q_1 + \lambda'_1 - a'_{1,1} + a'_{1,1} = q_1 + \lambda'_1$ , since  $\widehat{\mu}'_1 + \cdots + \widehat{\mu}'_{k+1} = \lambda'_1 - a'_{1,1}$ , and hence, the max  $y$ -deg of  $\widetilde{\Delta}_B(x, y)$  is  $2m - 1 + (\lambda_1 - a'_{1,1}) + s_1 + \lambda'_1 + q_1 + \sum_{i=2}^m (s_i + \lambda_i) + \sum_{j=2}^m (q_j + \lambda_j) = 2m - 1 + \sum_{i=1}^m s_i + \sum_{j=1}^m q_j + \lambda - a'_{1,1}$ . Since  $a'_{1,1} > 0$ , we cannot get a term of the max  $y$ -degree from  $\Delta(\widehat{S})$ .

Case 2.  $a_{1,s_1+1} < a'_{1,1}$  or  $a_{1,s_1+1} = a'_{1,1}$ . A similar argument works, and hence omit the details. Therefore, to evaluate  $\widetilde{f}_\nu(x)$ , it suffices to consider  $\widetilde{\Delta}(P_i)$ , since the treatment for  $\widetilde{\Delta}(Q_j)$  is similar to  $\widetilde{\Delta}(P_i)$ . In other words, we will show the following:

**Proposition 5.2.** *Let  $S = [[u_1, v_1, u_2, v_2, \dots, u_s, v_s, u_{s+1}]]$ , where  $u_i > 0, 1 \leq i \leq s + 1$ . Write  $\widetilde{\Delta}_B(x, y) = f_{s+\lambda}(x)y^{s+\lambda} + \cdots + f_0(x)$ , where  $\lambda = \sum_{i=1}^{s+1} u_i$ . Then we can write as follows, using some integer  $\gamma_{s+\lambda-\rho} \neq 0$ .*

$$f_{s+\lambda}(x) = \gamma_{s+\lambda-\rho} x^{s+\lambda-\rho} + \cdots + \gamma_\zeta x^\zeta, \text{ for some } \zeta \geq 0, s + \lambda - \rho > \zeta. \quad (5.16)$$

### 5.5. Auxiliary Lemmas

Before we proceed to the proof of Proposition 5.2, we show the following two lemmas.

**Lemma 5.3.** *Assume  $n \geq k \geq 0$  and  $n \geq m \geq 0$ . Then*

$$\begin{aligned} & \binom{n}{k} - \binom{n-1}{k-1} \binom{m}{m-1} + \binom{n-2}{k-2} \binom{m}{m-2} - \\ & \cdots + (-1)^\ell \binom{n-\ell}{k-\ell} \binom{m}{m-\ell} + \cdots + (-1)^m \binom{n-m}{k-m} \binom{m}{0} \\ & = \binom{n-m}{k} \end{aligned} \quad (5.17)$$

Note. In (5.17) we assume that  $\binom{n}{k} = 0$  if  $n \leq 0$  or  $k \leq 0$ , and  $\binom{0}{0} = 1$ .

*Proof.* We prove (5.17) by induction on  $n, k$  and  $m$ . Direct calculations prove the validity of the first step. Suppose (5.17) holds up to  $n, k$  and  $m - 1$ . Then we



see the following: The LHS of (5.17) is

$$\begin{aligned}
& \binom{n}{k} - \left\{ \binom{n-1}{k-1} \binom{m-1}{1} + \binom{n-1}{k-1} \binom{m-1}{0} \right\} \\
& + \left\{ \binom{n-2}{k-2} \binom{m-1}{2} + \binom{n-2}{k-2} \binom{m-1}{1} \right\} - \dots \\
& + (-1)^{m-1} \left\{ \binom{n-m+1}{k-m+1} \binom{m-1}{m-1} + \binom{n-m+1}{k-m+1} \binom{m-1}{m-2} \right\} \\
& + (-1)^m \left\{ \binom{n-m}{k-m} \binom{m-1}{m-1} \right\} \\
& = \binom{n}{k} - \binom{n-1}{k-1} \binom{m-1}{1} + \binom{n-2}{k-2} \binom{m-1}{2} + \dots \\
& + (-1)^{m-1} \binom{n-m+1}{k-m+1} \binom{m-1}{m-1} \\
& - \left\{ \binom{n-1}{k-1} \binom{m-1}{0} - \binom{n-2}{k-2} \binom{m-1}{1} + \dots \right. \\
& \left. + (-1)^{m-1} \binom{n-m}{k-m} \binom{m-1}{m-1} \right\} \\
& = \binom{n-(m-1)}{k} - \binom{n-m}{k-1} = \binom{n-m}{k} + \binom{n-m}{k-1} - \binom{n-m}{k-1} = \binom{n-m}{k},
\end{aligned}$$

by induction hypothesis.  $\square$

**Lemma 5.4.** *Let  $n \geq k \geq 0$ . Then the following equality holds among integer polynomials in  $n$  variables  $x_1, x_2, \dots, x_n$ :*

$$\begin{aligned}
& \binom{n}{k} x_1 \cdots x_n - \binom{n-1}{k} \sum_{\Lambda_{n-1}} x_{i_1} \cdots x_{i_{n-1}} \\
& + \binom{n-2}{k} \sum_{\Lambda_{n-2}} x_{i_1} \cdots x_{i_{n-2}} + \dots + (-1)^{n-k} \binom{k}{k} \sum_{\Lambda_k} x_{i_1} \cdots x_{i_k} \\
& = \binom{n}{k} (x_1 - 1) \cdots (x_n - 1) + \binom{n-1}{k-1} \sum_{\Lambda_{n-1}} (x_{i_1} - 1) \cdots (x_{i_{n-1}} - 1) \\
& + \binom{n-2}{k-2} \sum_{\Lambda_{n-2}} (x_{i_1} - 1) \cdots (x_{i_{n-2}} - 1) + \dots \\
& + \binom{n-k}{0} \sum_{\Lambda_{n-k}} (x_{i_1} - 1) \cdots (x_{i_{n-k}} - 1),
\end{aligned}$$

where the summation is taken over the set  $\Lambda_j$  consisting of all indices  $i_1, \dots, i_j$  such that  $1 \leq i_1 < \dots < i_j \leq n$ . If  $n-k=0$ , then the last term on the right side is interpreted as 1.

*Proof.* Since polynomials on both sides are symmetric polynomials over the symmetric group  $S_n$ , it is enough to compare the coefficients of  $x_1 x_2 \cdots x_m$ ,  $1 \leq m \leq n$ . For example, the constant term of the LHS is 0 if  $k > 0$ , while that of the RHS is

$$\begin{aligned}
& (-1)^n \binom{n}{k} + (-1)^{n-1} \binom{n-1}{k-1} \binom{n}{n-1} + \cdots + (-1)^{n-k} \binom{n-k}{0} \binom{n}{n-k} \\
&= (-1)^n \sum_{i=0}^k (-1)^i \binom{n-i}{k-i} \binom{n}{n-i} \\
&= (-1)^n \sum_{i=0}^k (-1)^i \binom{n}{k} \binom{k}{i} \\
&= (-1)^n \binom{n}{k} \sum_{i=0}^k (-1)^i \binom{k}{i} \\
&= 0
\end{aligned}$$

First,  $x_1 x_2 \cdots x_n$  appears  $\binom{n}{k}$  times in both sides. Thus the formula is true for  $x_1 x_2 \cdots x_n$ . Next, consider  $x_1 x_2 \cdots x_{n-1}$ . This appears  $-\binom{n-1}{k}$  times in the LHS, while it appears, in the RHS,  $-\binom{n}{k} + \binom{n-1}{k-1} = -\binom{n-1}{k}$  times. Thus the formula is true. In general,  $x_1 x_2 \cdots x_{n-r}$ ,  $r \geq 1$ , appears  $(-1)^r \binom{n-r}{k}$  times in the LHS, while in the RHS, it appears as many times as

$$\begin{aligned}
& (-1)^r \left\{ \binom{n}{k} - \binom{n-1}{k-1} \binom{r}{r-1} + \binom{n-2}{k-2} \binom{r}{r-2} - \cdots + (-1)^r \binom{n-r}{k-r} \right\} \\
&= (-1)^r \binom{n-r}{k} \text{ (by Lemma 5.3).}
\end{aligned}$$

□

### 5.6. Proof of Proposition 5.2.

By (5.10), we can write

$$\begin{aligned}
\tilde{\Delta}_B(x, y) &= (xy - 1)^{s+1} \Delta_B(x, y) \\
&= \sum_{p=0}^s \sum_{\Lambda_{p,s}} (-1)^p v_{i_1} v_{i_2} \cdots v_{i_p} (x-1)^p (y-1)^p \tilde{F}_{\mu_1} \tilde{F}_{\mu_2} \cdots \tilde{F}_{\mu_{p+1}} (xy-1)^{s-p},
\end{aligned} \tag{5.18}$$

where  $\mu_1 + \cdots + \mu_{p+1} = \lambda = u_1 + u_2 + \cdots + u_{s+1}$ .

In (5.18), terms with  $y^{s+\lambda}$  are obtained as follows. Let  $B_k$  be the coefficient of the term  $x^{s+\lambda-k} y^{s+\lambda}$ .

(1) For  $p = 0$ , since we smooth all crossings at  $v_i$ , we have only one term  $\tilde{F}_{\mu_1} (xy-1)^s$ . Since  $\mu_1 = \lambda$ , we have one term  $x^{\lambda+s} y^{\lambda+s}$ .

(2) For  $p = 1$ , we have the following polynomial

$$(-1) \sum_{i=1}^s v_i (x-1)(y-1) \tilde{F}_{\mu_1} \tilde{F}_{\mu_2} (xy-1)^{s-1}.$$

Thus the contribution to  $B_0$  by these polynomials is  $(-1) \sum_{i=1}^s v_i$ .

(3) For general  $p$ , the contribution to  $B_0$  by the polynomials in

$$\begin{aligned} & (-1)^p \sum v_{i_1} v_{i_2} \cdots v_{i_p} (x-1)^p (y-1)^p \tilde{F}_{\mu_1} \tilde{F}_{\mu_2} \cdots \tilde{F}_{\mu_{p+1}} (xy-1)^{s-p} \text{ is} \\ & (-1)^p \sum_{1 \leq i_1 < \cdots < i_p \leq s} v_{i_1} v_{i_2} \cdots v_{i_p}. \end{aligned}$$

Therefore, by letting  $n = s$  and  $k = 0$  in Lemma 5.4, we have:

$$\begin{aligned} B_0 &= 1 - \sum_{i=1}^s v_i + \sum_{1 \leq i < j \leq s} v_i v_j + \cdots \\ &\quad + (-1)^p \sum_{\Lambda_{p,s}} v_{i_1} v_{i_2} \cdots v_{i_p} + \cdots + (-1)^s v_1 v_2 \cdots v_s \\ &= (-1)^s (v_1 - 1)(v_2 - 1) \cdots (v_s - 1). \end{aligned} \tag{5.19}$$

If  $\rho = 0$ , i.e.,  $v_j \neq 1$  for any  $j$ , then  $B_0 \neq 0$ , i.e.  $x^{\lambda+s} y^{\lambda+s}$  does exist. However, if  $\rho > 0$ , then  $B_0 = 0$ , and hence  $x^{\lambda+s} y^{\lambda+s}$  does not exist. Next we consider  $B_1$ . The terms  $x^{\lambda+s-1} y^{\lambda+s}$  are obtained as follows.

(1) If  $p = 0$ , we do not get the term  $x^{\lambda+s-1} y^{\lambda+s}$ .

(2) Suppose  $p \geq 1$ . Then in order to get  $x^{\lambda+s-1} y^{\lambda+s}$ , we must take every possible  $y$ -term of maximal degree. In other words, from each  $\tilde{F}_{\mu_j}$ , take  $(xy)^{\mu_j}$  and  $(xy)^{s-p}$  from  $(xy-1)^{s-p}$  and  $y^p$  from  $(y-1)^p$ . For the  $x$ -terms we take  $(-1) \binom{p}{p-1} x^{p-1}$  from  $(x-1)^p$ .

Therefore, we have, by Lemma 5.4,

$$\begin{aligned} B_1 &= (-1) \sum_{1 \leq i_1 \leq s} v_{i_1} (-1) \binom{1}{0} + (-1)^2 \sum_{\Lambda_{2,s}} v_{i_1} v_{i_2} (-1) \binom{2}{1} \\ &\quad + (-1)^3 \sum_{\Lambda_{3,s}} v_{i_1} v_{i_2} v_{i_3} (-1) \binom{3}{2} + \cdots + (-1)^s v_1 \cdots v_s (-1) \binom{s}{s-1} \\ &= (-1) \left\{ s(v_1 - 1) \cdots (v_s - 1) + \sum_{1 \leq i_1 < \cdots < i_{s-1} \leq s} (v_{i_1} - 1) \cdots (v_{i_{s-1}} - 1) \right\}. \end{aligned} \tag{5.20}$$

If  $\rho = 1$ , then the first term in the RHS is 0, but one term in the second summation survives. Thus,  $x^{\lambda+s-1} y^{\lambda+s}$  does exist, and

$$B_1 = -(v_1 - 1)(v_2 - 1) \cdots (v_{t-1} - 1)(v_{t+1} - 1) \cdots (v_s - 1) \text{ for some } t.$$

However, if  $\rho \geq 2$ , then  $B_1 = 0$ . By the same argument, we can show;

$$\begin{aligned}
B_r &= (-1)^r \sum_{\Lambda_{r,s}} v_{i_1} \cdots v_{i_r} (-1)^r \binom{r}{0} + (-1)^{r+1} \sum_{\Lambda_{r+1,s}} v_{i_1} \cdots v_{i_{r+1}} (-1)^r \binom{r+1}{1} \\
&\quad + (-1)^{r+2} \sum_{\Lambda_{r+2,s}} v_{i_1} \cdots v_{i_{r+2}} (-1)^r \binom{r+2}{2} + \cdots + (-1)^s v_1 \cdots v_s (-1)^r \binom{s}{s-r} \\
&= (-1)^{s+r} \left\{ \binom{s}{r} v_1 \cdots v_s - \binom{s-1}{r} \sum_{\Lambda_{s-1,s}} v_{i_1} \cdots v_{i_{s-1}} \right. \\
&\quad \left. + \binom{s-2}{r} \sum_{\Lambda_{s-2,s}} v_{i_1} \cdots v_{i_{s-2}} - \cdots \right. \\
&\quad \left. + (-1)^{s-r-1} \binom{r+1}{r} \sum_{\Lambda_{r+1,s}} v_{i_1} \cdots v_{i_{r+1}} + (-1)^{s-r} \binom{r}{r} \sum_{\Lambda_{r,s}} v_{i_1} \cdots v_{i_r} \right\} \\
&= (-1)^{s+r} \left\{ \binom{s}{r} (v_1 - 1) \cdots (v_s - 1) + \binom{s-1}{r-1} \sum_{\Lambda_{s-1,s}} (v_{i_1} - 1) \cdots (v_{i_{s-1}} - 1) + \cdots \right. \\
&\quad \left. + \binom{s-r}{0} \sum_{\Lambda_{s-r,s}} (v_{i_1} - 1) \cdots (v_{i_{s-r}} - 1) \right\} \tag{5.21}
\end{aligned}$$

Thus, if  $\rho \geq r+1$ , then  $B_r = 0$ . However, if  $\rho = r$ , say  $v_1 = v_2 = \cdots = v_r = 1$ , but  $v_j \neq 1, j \geq r+1$ , then only the last summation contains one non-zero term:  $(v_{r+1} - 1) \cdots (v_s - 1) \neq 0$ . Therefore, if  $\rho = r$ , then  $B_0 = B_1 = \cdots = B_{\rho-1} = 0$ , but there exist  $s - \rho$  integers  $v_{i_1}, v_{i_2}, \cdots, v_{i_{s-\rho}}$ , each of which is not 1, and

$$B_\rho = (-1)^{s+r} (v_{i_1} - 1) (v_{i_2} - 1) \cdots (v_{i_{s-\rho}} - 1) \neq 0. \tag{5.22}$$

This proves Proposition 5.2. □

### 5.7. Precise form of $\Delta_B(x, y)$

Now we arrive at our final theorem of this section.

**Theorem 5.5.** *Let  $S = \{P_1, d_1, Q_1, e_1, P_2, d_2, Q_2, e_2, \cdots, P_m, d_m, Q_m\}$  be the canonical decomposition of the continued fraction of  $\beta/2\alpha$ . Let  $\rho$  and  $\lambda$  be the numbers defined in Definition 3.10. Write*

$$\Delta_{B(2\alpha, \beta)}(x, y) = f_{\lambda-1}(x) y^{\lambda-1} + \cdots + f_0(x),$$

where  $f_{\lambda-1}(x) \neq 0$  and  $f_0(x) \neq 0$ , and  $f_i(x), 0 \leq i \leq \lambda-1$ , are integer polynomials.

Then we have:

- (1)  $f_i(x^{-1})x^{\lambda-1} = f_{\lambda-1-i}(x), 0 \leq i \leq \lambda-1,$
- (2)  $f_{\lambda-1}(x) = \gamma_{\lambda-1, \lambda-1-\rho}x^{\lambda-1-\rho} + \cdots + \gamma_{\lambda-1, \zeta}x^{\zeta},$   
 $f_{\lambda-2}(x) = \gamma_{\lambda-2, \lambda-1-\rho+1}x^{\lambda-1-\rho+1} + \cdots,$   
 $\dots$   
 $f_{\lambda-i-1}(x) = \gamma_{\lambda-i-1, \lambda-1-\rho+i}x^{\lambda-1-\rho+i} + \cdots$   
 $\dots$   
 $f_{\lambda-1-\rho}(x) = \gamma_{\lambda-1-\rho, \lambda-1}x^{\lambda-1} + \cdots,$  where  $\gamma_{\lambda-1, \lambda-1-\rho} = \gamma_{\lambda-1-\rho, \lambda-1} \neq 0,$   
and hence,  
 $\deg f_{\lambda-1}(x) = \lambda-1-\rho, \deg f_{\lambda-1-i}(x) \leq \lambda-1+i-\rho, 1 \leq i \leq \rho-1$  and  
 $\deg f_{\lambda-1-\rho}(x) = \lambda-1.$
- (3) All non-zero leading coefficients of  $f_i(x)$  are of the same sign.
- (4)  $\prod_{i=1}^m d_i \prod_{j=1}^{m-1} e_j$  divides  $\gamma_{\lambda-1, \lambda-1-\rho}$  (and  $\gamma_{\lambda-1-\rho, \lambda-1}$ )
- (5)  $\gamma_{\lambda-1, \lambda-1-\rho}$  (and  $\gamma_{\lambda-1-\rho, \lambda-1}$ ) is equal to  $\pm 1$  if and only if
  - (i) all  $d_i = \pm 1, 1 \leq i \leq m,$  and
  - (ii) all  $e_j = \pm 1, 1 \leq j \leq m-1,$  and
  - (iii) all  $b_{i,k}, 1 \leq i \leq m, 1 \leq k \leq s_i$   
and all  $b'_{j,k}, 1 \leq j \leq m, 1 \leq k \leq q_j$  are either 1 or 2.

(5.23)

*Proof.* (1) Since a 2-bridge link  $B(2\alpha, \beta)$  is invertible, we have  $\Delta_{B(2\alpha, \beta)}(x^{-1}, y^{-1})x^{\lambda-1}y^{\lambda-1} = \Delta_{B(2\alpha, \beta)}(x, y)$ . This implies:

$$x^{\lambda-1}y^{\lambda-1} \left\{ f_{\lambda-1}(x^{-1})y^{-(\lambda-1)} + f_{\lambda-2}(x^{-1})y^{-(\lambda-2)} + \cdots + f_0(x^{-1}) \right\} \\ = f_{\lambda-1}(x^{-1})x^{\lambda-1} + f_{\lambda-2}(x^{-1})x^{\lambda-1}y + \cdots + f_0(x^{-1})x^{\lambda-1}y^{\lambda-1}, \text{ and hence, we} \\ \text{have (1).}$$

(2) Proposition 5.2 shows that  $f_{\lambda-1}(x)$  is a required form. Since  $B(2\alpha, \beta)$  is interchangeable, we see  $\Delta_B(x, y) = \Delta_B(y, x)$ , and hence  $\gamma_{\lambda-1, \lambda-1-\rho} = \gamma_{\lambda-1-\rho, \lambda-1}$ .

Next, to show that  $\deg f_{\lambda-1-i} \leq \lambda-1+i-\rho, 1 \leq i \leq \rho-1$ , we need the following easy lemma.

**Lemma 5.6.** *The number of terms of  $\Delta_{B(2\alpha, \beta)}(x, y)$  is exactly  $\alpha$ . In other words, if we write  $\Delta_{B(2\alpha, \beta)}(x, y) = \sum_{0 \leq p, q} c_{p, q} x^p y^q$ , then  $\sum_{0 \leq p, q} |c_{p, q}| = \alpha$ .*

*Proof.* The group of  $B(2\alpha, \beta)$  has the following Wirtinger presentation:  $\pi_1(S^3 - B(2\alpha, \beta)) = \langle x, y | R \rangle$ , where  $R = WxW^{-1}x^{-1}$ , and  $W = y^{\varepsilon_1}x^{\varepsilon_2}y^{\varepsilon_3} \cdots y^{\varepsilon_{2\alpha-1}}, \varepsilon_i = \pm 1$ . Therefore, the Alexander matrix  $M$  is of the form:

$M = \left[ \frac{\partial R}{\partial x} \frac{\partial R}{\partial y} \right]^\phi = \left[ \frac{\partial W}{\partial x}(1-x) + W - 1, \frac{\partial W}{\partial y}(1-x) \right]^\phi = \left[ \frac{\partial W}{\partial y}(1-y) \frac{\partial W}{\partial y}(1-x) \right]^\phi$   
 and hence,  $\Delta_B(x, y) = \det \left[ \frac{\partial W}{\partial y} \right]^\phi$ .

Here  $\det \left[ \frac{\partial W}{\partial y} \right]^\phi$  is the sum of  $\alpha$  terms, while  $|\Delta_{B(2\alpha, \beta)}(-1, -1)| = \alpha$ , and hence no cancellation occurs among these  $\alpha$  terms.  $\square$

Now we return to the proof of (2). Suppose  $\deg f_{\lambda-1-i}(x) > \lambda - 1 + i - \rho$ . Then  $\deg f_{\lambda-1-i}(t)t^{\lambda-1-i} > 2(\lambda - 1) - \rho$ .

Write  $f_{\lambda-1-i}(x) = \gamma_{\lambda-1-i,k}x^k + \cdots + \gamma_{\lambda-1-i,r}x^r$ , where  $k > \lambda - 1 + i - \rho$ , and  $k \geq r$ . Then by (1),

$$f_i(x) = f_{\lambda-1-i}(x^{-1})x^{\lambda-1} = \gamma_{\lambda-1-i,r}x^{\lambda-1-r} + \cdots + \gamma_{\lambda-1-i,k}x^{\lambda-1-k}.$$

Since  $\lambda - 1 - r \geq \lambda - 1 - k$ ,  $\Delta_B(t, t)$  contains the term with degree  $\lambda - 1 - k + i$ . Since no cancellation occurs when we set  $x = y = t$ , we see

$$\begin{aligned} \deg \Delta_{B(2\alpha, \beta)}(t, t) &> 2(\lambda - 1) - \rho - (\lambda - 1 - k + i) \\ &= \lambda - 1 - i - \rho + k \\ &> \lambda - 1 - i - \rho + \lambda - 1 + i - \rho \\ &= 2\lambda - 2 - 2\rho. \end{aligned}$$

This contradicts Proposition 3.13. This proves (2).

(3) follows also from the fact that no cancellations occur when we set  $x = y = t$  in  $\Delta_B(x, y)$ . (4) follows from (5.14). (5) follows also from (5.14) and (5.22).

Theorem 5.5 is now proved.  $\square$

**Remark 5.7.** It is quite likely that

$$\deg f_{\lambda-1-i}(x) = \lambda - 1 + i - \rho, 1 \leq i \leq \rho - 1. \quad (5.24)$$

## 6. Monic Alexander polynomials

In this section, we determine when the Alexander polynomial of  $K(2\alpha, \beta|r)$ ,  $r > 0$ , is monic. We use the results proved in the previous section. In Subsection 6.1, we deal with the case  $\ell k B(2\alpha, \beta) \neq 0$ , using the continued fraction of  $\beta/2\alpha$ . However, if  $\ell k B(2\alpha, \beta) = 0$ , we cannot characterize  $K(2\alpha, \beta|r)$  with monic Alexander polynomials in terms of continued fractions. We then deal with this case in Subsection 6.2. Let  $\{P_1, d_1, Q_1, e_1, P_2, d_2, Q_2, e_2, \dots, P_m, d_m, Q_m\}$  be the canonical decomposition of the continued fraction of  $\beta/2\alpha$ .

Write  $P_i = [[a_{i,1}, b_{i,1}, a_{i,2}, b_{i,2}, \dots, a_{i,s_i}, b_{i,s_i}, a_{i,s_i+1}]]$ ,  $a_{i,j} > 0$ , and  $Q_j = [[-a'_{j,1}, -b'_{j,1}, \dots, -a'_{j,q_j}, -b'_{j,q_j}, -a'_{j,q_j+1}]]$ ,  $a'_{j,k} > 0$ .

### 6.1. The case $\ell k B(2\alpha, \beta) > 0$ .

The purpose of this subsection is to state algebraic conditions equivalent to that in Theorem 2.2. Namely, we prove the following:

**Theorem 6.1.** Suppose  $\ell = \ell k B(2\alpha, \beta) \neq 0$ .

- (1) Suppose  $\ell = r = 1$ . Then,  $\Delta_{K(2\alpha, \beta|1)}(t)$  is monic if and only if, for any  $i, j, p, q$ ,  
 (a)  $d_i, e_j = \pm 1$  and (b)  $b_{i,k} = b'_{j,p} = 2$ ,  
 (2) Suppose  $\ell \geq 2$ . Then for any  $r \geq 1$ ,  $\Delta_{K(2\alpha, \beta|r)}(t)$  is monic if and only if  
 (a)  $d_i, e_j = \pm 1$  and (b)  $b_{i,k}$  and  $b'_{j,p}$  are 1 or 2.

Let  $\Delta_{B(2\alpha, \beta)}(x, y)$  be the Alexander polynomial of  $B(2\alpha, \beta)$ . Suppose  $\ell = \ell k B(2\alpha, \beta) > 0$ . Then by Proposition 4.1 (1) and Theorem 5.5, the Alexander polynomial  $\Delta_K(t)$  of  $K = K(2\alpha, \beta|r)$ ,  $r > 0$ , is given by

$$\Delta_K(t) = \frac{1-t}{1-t^\ell} \{ f_{\lambda-1}(t)t^{(\lambda-1)\ell r} + f_{\lambda-2}(t)t^{(\lambda-2)\ell r} + \cdots + f_0(t) \}, \quad (6.1)$$

where  $\lambda - 1$  is the maximal  $y$ -degree of  $\Delta_{B(2\alpha, \beta)}(x, y)$ .

First we determine the degree of  $\Delta_{B(2\alpha, \beta)}(t, t^{\ell r})$ .

**Proposition 6.2.** (1) The highest degree of  $\Delta_{B(2\alpha, \beta)}(t, t^{\ell r})$  is

$$\lambda - 1 - \rho + (\lambda - 1)\ell r. \quad (6.2)$$

(2) The lowest degree of  $\Delta_{B(2\alpha, \beta)}(t, t^{\ell r})$  is  $\rho$ .

*Proof.* (1) We write  $f_{\lambda-1}(x) = \gamma_{\lambda-1, \lambda-1-\rho} x^{\lambda-1-\rho} + \cdots + \gamma_{\lambda-1, \zeta} x^\zeta$ ,  $\gamma_{\lambda-1, \lambda-1-\rho} \neq 0$ . We show that if  $\ell r \geq 2$ , then  $\gamma_{\lambda-1, \lambda-1-\rho} t^{\lambda-1-\rho} t^{(\lambda-1)\ell r}$  is the only term with the highest degree in  $\Delta_{B(2\alpha, \beta)}(t, t^{\ell r})$ . In fact, by Theorem 5.5, we see that for  $1 \leq i \leq \rho - 1$ ,  $\deg f_{\lambda-1-i}(x) \leq \lambda - 1 + i - \rho$ , and hence, since  $\ell r \geq 1$ , for  $1 \leq i \leq \rho - 1$  we have:

$$\lambda - 1 - \rho + (\lambda - 1)\ell r \geq \lambda - 1 + i - \rho + (\lambda - 1 - i)\ell r. \quad (6.3)$$

Moreover, obviously,  $\deg f_j(x) \leq \lambda - 1$ , for  $0 \leq j \leq \lambda - 2 - \rho$ , and hence, if  $\ell r \geq 1$ , then for  $0 \leq j \leq \lambda - 2 - \rho$ , we see

$$\lambda - 1 - \rho + (\lambda - 1)\ell r \geq \lambda - 1 + j\ell r. \quad (6.4)$$

Combining (6.3) and (6.4), we have (1). In particular, if  $\ell r \geq 2$ , the strict inequality holds in (6.3), and therefore,  $\gamma_{\lambda-1, \lambda-1-\rho} t^{\lambda-1-\rho} t^{(\lambda-1)\ell r}$  is the only term with the highest degree in  $\Delta_{B(2\alpha, \beta)}(t, t^{\ell r})$ . However, if  $\ell r = 1$ , then the equality holds in (6.3), and thus, the terms with the highest degree appear at least in  $f_{\lambda-1}(t)t^{(\lambda-1)\ell r}$  and  $f_{\lambda-1-\rho}(t)t^{(\lambda-1-\rho)\ell r}$ .

(2) First we note from Proposition 3.13 that the lowest degree of  $\Delta_{B(2\alpha, \beta)}(t, t)$  is  $\rho$ , since the highest degree of  $\Delta_{B(2\alpha, \beta)}(t, t)$  is  $2(\lambda - 1) - \rho$  by (1).

By Theorem 5.5, we see that  $\deg f_{\lambda-1-i}(x) \leq \lambda - 1 + i - \rho$ ,  $1 \leq i \leq \rho - 1$  and, of course,  $\deg f_j \leq \lambda - 1$ , for  $0 \leq j \leq \lambda - 2 - \rho$ .

Now, since  $f_0(x) = f_{\lambda-1}(x^{-1})x^{\lambda-1}$ , it follows that the lowest degree of  $f_{\lambda-1}(t)t^{(\lambda-1)\ell r}$  is  $\zeta + (\lambda - 1)\ell r$ , and that of  $f_0(t)$  is  $\rho$ . Since  $\rho \leq \lambda - 1$ ,  $\min\{\zeta + (\lambda - 1)\ell r, \rho\} = \rho$ , and hence, if  $\ell r \geq 1$ ,  $f_0(t)$  contains the term of degree  $\rho$ . Furthermore, since the lowest degree of  $\Delta_{B(2\alpha, \beta)}(t, t)$  is  $\rho$ , the degree of any term in  $\Delta_{B(2\alpha, \beta)}(t, t^{\ell r})$  is at least  $\rho$ . This proves (2).  $\square$

Proposition 6.2 implies the following:

**Proposition 6.3.** *If  $\ell r \geq 1$ , then  $\deg \Delta_{K(r)}(t) = (\lambda - 1)(\ell r + 1) - 2\rho - (\ell - 1)$ . In particular, if  $\ell r = 1$  (i.e.  $\ell = r = 1$ ), then  $\deg \Delta_{K(r)}(t) = 2(\lambda - \rho - 1)$ .*

Using the above results, we can characterize the monic Alexander polynomial.

First, we see that if  $\ell r \geq 2$ , then the leading coefficient of  $\Delta_K(t)$  is given by  $\gamma_{\lambda-1, \lambda-1-\rho}$ . Therefore,  $\Delta_K(t)$  is monic if and only if  $\gamma_{\lambda-1, \lambda-1-\rho}$  is  $\pm 1$ , and hence Theorem 5.5(5) gives us immediately the following:

**Proposition 6.4.** *Suppose  $\ell r \geq 2$ . Then  $\Delta_K(t)$  is monic if and only if the following conditions hold:*

- (1)  $d_i, e_j = \pm 1$  for any  $i, j$ , and
- (2)  $b_{i,k} = 1$  or  $2$  and  $b'_{j,p} = 1$  or  $2$ , for any  $1 \leq i \leq m$ ,  $1 \leq k \leq s_i + 1$ , and  $1 \leq j \leq m$ ,  $1 \leq p \leq q_j + 1$ .

Note that  $a_{i,j}$  and  $a'_{j,k}$  are arbitrary.

If  $\ell r = 1$ , then the following proposition holds.

**Proposition 6.5.** *Suppose  $\ell = r = 1$ . Then  $\Delta_{K(1)}(t)$  is monic if and only if*

- (1)  $d_i, e_j = \pm 1$  for any  $i, j$ , and
- (2)  $b_{i,k} = b'_{j,p} = 2$  for any  $i, k, j, p$ . (In particular,  $\rho = 0$ .)

*Proof.* (1) Suppose that  $d_i$  or  $e_j$  is not  $\pm 1$ . Then  $\Delta_{K(1)}(t)$  is not monic by Theorem 5.5. (2) Suppose  $\rho \neq 0$ . Then  $\Delta_{B(2\alpha, \beta)}(x, y)$  contains at least two non-zero terms,  $\gamma_{\lambda-1, \lambda-1-\rho} x^{\lambda-1-\rho} y^{\lambda-1}$  and  $\gamma_{\lambda-1-\rho, \lambda-1} x^{\lambda-1} y^{\lambda-1-\rho}$ . Since  $\gamma_{\lambda-1, \lambda-1-\rho} = \gamma_{\lambda-1-\rho, \lambda-1}$  by Theorem 5.5(2), we see that  $\Delta_{K(2\alpha, \beta|1)}(t)$  is not monic. Further, as is proved in Subsection 5.6, we have all  $b_{i,k} = b'_{j,k} = 2$ . The converse follows from Theorem 5.5.  $\square$

By these results above, we obtain Theorem 6.1.

Finally, we note that we can prove the following (c.f., [15, Theorem 4.2]) as a simple consequence of Proposition 6.3.

**Proposition 6.6.** *Suppose  $\ell, r > 0$ . Then  $K(2\alpha, \beta|r)$  is unknotted if and only if  $(2\alpha, \beta) = (4, 3)$  and  $r = 1$ , and hence  $\ell = 2$ .*

*Proof.* Since the “if” part is obvious, we only consider the “only if” part. Suppose  $K(r) = K(2\alpha, \beta|r)$  is unknotted. Then, by Proposition 6.3, we have:

$$(\lambda - 1)(\ell r + 1) - 2\rho - (\ell - 1) = 0. \quad (6.5)$$

Rewrite the LHS of (6.5) as

$$2(\lambda - 1 - \rho) + (\lambda - 1)(r - 1)\ell + (\lambda - 2)(\ell - 1) = 0. \quad (6.6)$$

Since  $\lambda \geq 2$  and  $\lambda - 1 \geq \rho$ , it follows that each term of the LHS is non-negative, and hence, the equality holds only if we have: (i)  $\lambda - 1 = \rho$ ,  $r = 1$ , and (ii)  $\lambda = 2$  or



$\ell = 1$ . Now, since  $\lambda - 1 = \rho$ , we see that  $(2\alpha, \beta) = (2\lambda, 2\lambda - 1)$  and  $\ell k B(2\alpha, \beta) = \lambda$ . Since  $\lambda \geq 2$ ,  $\ell$  cannot be 1, and hence the conclusion follows.  $\square$

### 6.2. The case $\ell k B(2\alpha, \beta) = 0$ .

In this subsection, we characterize the monic Alexander polynomial of  $K(r)$  when  $\ell k B(2\alpha, \beta) = 0$ . To do this, first, we give a geometric interpretation of  $\Delta_{K(r)}(t)$ . We remind that the calculation of the Alexander polynomial of  $K(r)$  from  $\Delta_{B(2\alpha, \beta)}(x, y)$  is quite different for  $\ell k B(2\alpha, \beta) = 0$ , as is seen in Proposition 4.1. Let  $B(2\alpha, \beta)$  be a 2-bridge link consisting of  $K_1$  and  $K_2$ . Suppose that  $\ell k(K_1, K_2) = 0$ . Consider the infinite cyclic cover  $M^3$  of  $S^3 \setminus K_2$ . Since  $K_2$  is unknotted,  $M^3$  is an infinite cylinder  $D^2 \times \mathbb{R}^1$ . Let  $\{\tilde{K}_m, m = 0, \pm 1, \pm 2, \dots\}$  be the set of lifts of  $K_1$  in  $M^3$ , where  $\tilde{K}_j = \psi^j(\tilde{K}_0)$  with the covering translation  $\psi$ . Since  $\ell k(K_1, K_2) = 0$ , each lift  $\tilde{K}_m$  is a knot in  $M^3$  with orientation inherited from that of  $K_1$ . Denote  $c_j = \ell k(\tilde{K}_0, \tilde{K}_j)$ ,  $j \neq 0$ , and let

$$\Gamma(t) = \sum_{-\infty < j < \infty} c_j t^j, \text{ where } c_0 = - \sum_{-\infty < j < \infty} c_j. \quad (6.7)$$

Note that  $c_0$  is well defined, and also  $c_j = c_{-j}$  for any  $j$ . Then it is proved in [16]

**Proposition 6.7.**  $\Gamma(t) \doteq (t - 1) \left[ \frac{\Delta_B(x, y)}{1 - y} \right]_{x=t, y=1}$

We note that Gonzalez-Acuña also studied this polynomial  $\Gamma(t)$  in [9].

Using Proposition 6.7, we can estimate the maximal and minimal degree of  $\Delta_{K(2\alpha, \beta|r)}(t)$ . Let  $S = [[u_1, v_1, u_2, v_2, \dots, u_s, v_s, u_{s+1}]]$  be the continued fraction of  $\beta/2\alpha$ . Let  $G(S)$  be the graph of  $S$ . This graph  $G(S)$  will be used to estimate the degree of  $\Delta_{K(2\alpha, \beta|r)}(t)$ .

**Proposition 6.8.** *Let  $h$  and  $q$  be the highest and lowest  $y$ -coordinates of  $G(S)$ . Then  $\deg \Delta_{K(2\alpha, \beta|r)}(t) \leq 2 \max\{h, |q|\}$ .*

Note that  $h \geq 0$  and  $q \leq 0$ .

*Proof.* We span  $K_2$  by a disk  $D$  in such a way that  $K_1$  intersects  $D$  transversally at  $\sum_{i=1}^{s+1} |u_i|$  points. Using  $D$ , we construct  $M^3$ . Then it is easy to evaluate the linking number between  $\tilde{K}_0$  and  $\tilde{K}_j$  for  $j \geq 1$  using the primitive disk for  $K_1$ . See Example 12.1.

Also we can easily determine  $h$  and  $q$  from the graph  $G(S)$ . In fact,  $h$  is the  $y$ -coordinate of the absolute maximal vertices of  $G(S)$ , and  $q$  is the  $y$ -coordinate of the absolute minimal vertices. Let  $V_{i,1}, V_{i,2}, \dots, V_{i,p}$  be the absolute maximal vertices and  $V_{j,1}, V_{j,2}, \dots, V_{j,s}$  be the absolute minimal vertices of  $G(S)$ .

Let  $w_{i,k}$  be the weight of  $V_{i,k}$ ,  $k = 1, 2, \dots, p$ , and  $w_{j,n}$  the weight of  $V_{j,n}$ ,  $n =$

$1, 2, \dots, s$ . Then we have:

$$\begin{aligned}
(1) \text{ If } h > |q|, \text{ then } \ell k(\tilde{K}_0, \tilde{K}_h) &= -\frac{1}{2} \sum_{k=1}^p w_{i,k}. \\
(2) \text{ If } h < |q|, \text{ then } \ell k(\tilde{K}_0, \tilde{K}_q) &= -\frac{1}{2} \sum_{n=1}^s w_{j,n}. \\
(3) \text{ If } h = |q|, \text{ then } \ell k(\tilde{K}_0, \tilde{K}_h) &= -\frac{1}{2} \left\{ \sum_{k=1}^p w_{i,k} + \sum_{n=1}^s w_{j,n} \right\}. \\
(4) \text{ If } d > \max\{h, |q|\}, \text{ then } \ell k(\tilde{K}_0, \tilde{K}_d) &= 0.
\end{aligned} \tag{6.8}$$

Therefore,  $\max \deg \Gamma(t) \leq \max\{h, |q|\}$ , and  $\min \deg \Gamma(t) \geq -\max\{h, |q|\}$ , and hence,  $\deg \Delta_{K(2\alpha, \beta|r)}(t) \leq 2 \max\{h, |q|\}$ .  $\square$

Under the same notation used in the proof of Proposition 6.8, we have

**Corollary 6.9.**  $\left[ \frac{\Delta_{K(2\alpha, \beta)}(x, y)}{1 - y} \right]_{\substack{x=t \\ y=1}} (1 - t)$  is monic and its degree is equal to  $2 \max\{h, |q|\}$  if and only if

- (1)  $\sum_{k=1}^p w_{i,k} = \pm 2$ , when  $h > |q|$ ,
- (2)  $\sum_{n=1}^s w_{j,n} = \pm 2$ , when  $h < |q|$ ,
- (3)  $\sum_{k=1}^p w_{i,k} + \sum_{n=1}^s w_{j,n} = \pm 2$ , when  $h = |q|$ .

**Proposition 6.10.** Suppose  $\ell k B(2\alpha, \beta) = 0$ . Then, for  $r > 1$ ,

- (1)  $\Delta_{K(2\alpha, \beta|r)}(t)$  is either non-monic or  $\Delta_{K(2\alpha, \beta|r)}(t) = 1$ .
- (2)  $\Delta_{K(2\alpha, \beta|1)}(t)$  is monic if and only if  $\left[ \frac{\Delta_{K(2\alpha, \beta)}(x, y)}{1 - y} \right]_{\substack{x=t \\ y=1}}$  is monic.

This is an immediate consequence of Proposition 4.1.

The rest of this paper (except for the last three sections) will be devoted to the proofs of our main theorems.

## 7. Construction of a Seifert surface $F_1$ for $K_1$ .

By Theorem 3.14, we assume  $\ell k B(2\alpha, \beta) = \ell \geq 0, r > 0$ . In section 3, we constructed a primitive spanning disk  $F_D$  for  $K_1$ , which consists of disks and bands corresponding to the edges and vertices in  $G(S)$ . Recall that  $F_D$  intersects  $K_2$  as many times as the number of edges in  $G(S)$ , which is equal to  $\lambda$ . In this section, we construct a Seifert surface  $F(r)$  for  $K(2\alpha, \beta|r) = K(r)$ . First, using  $G(S)$ , we construct a new Seifert surface  $F_1$  for  $K_1$  which intersects  $K_2$  exactly  $\ell \geq 0$  times. We call  $F_1$  a *canonical surface* for  $K_1$ .

Let  $S = \{P_1, d_1, Q_1, e_1, P_2, \dots\}$  be the canonical decomposition of the continued fraction of  $\beta/2\alpha$ . Let  $G(S)$  be the graph of  $S$ , which by definition is the graph  $G(S^*)$  of the modified continued fraction  $S^*$  of  $S$ . We construct a new surface  $F_1$

by induction on  $\nu(G(S))$ , the total number of local maximal and local minimal vertices, including the end vertices.

Case 1:  $\nu(G) = 2$

Since  $\ell \geq 0$ ,  $G$  is an ascending line segment. See Figure 7.1 (a).

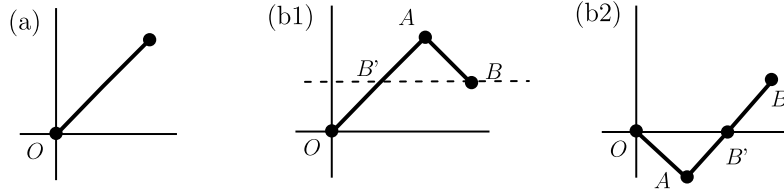


Figure 7.1: Graphs  $G$  for  $\nu(G) = 2$  and 3

In this case,  $F_D$  itself is our new surface  $F_1$ , which consists of two disks connected by a twisted band.

Case 2:  $\nu(G) = 3$

There are two cases. See Figure 7.1 (b1) and (b2). Note that the vertex  $B$  may be on the  $x$ -axis.

For the first case (b1),  $F_D$  consists of  $p$ , say, positive disks  $D_1, D_2, \dots, D_p$  followed by  $q$ , say, negative disks  $D'_1, D'_2, \dots, D'_q$ ,  $p \geq q$ , and  $p + q - 1$  bands  $B_j, j = 1, 2, \dots, p + q - 1$ , connecting these disks. See Figure 7.2 (a).

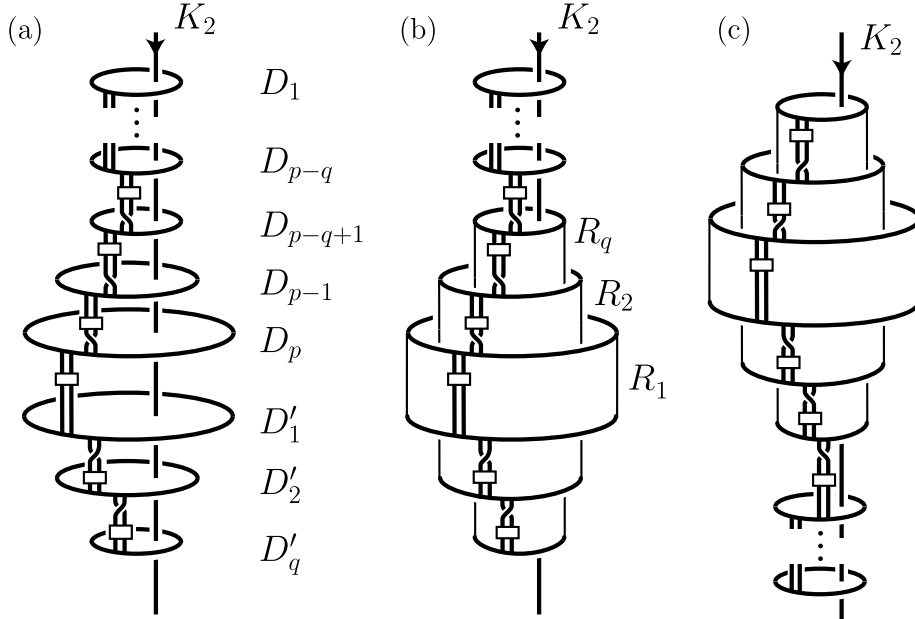


Figure 7.2: Construction of a canonical surface

Then replace two disks  $D_p$  and  $D'_1$  by a cylinder  $R_1$ , where  $R_1 \cap F_D = \partial R_1 = \partial(D_p \cup D'_1)$ . The orientation of  $R_1$  is naturally induced from those of disks. Next we replace two disks  $D_{p-1}$  and  $D'_2$  by a cylinder  $R_2$  that is inside of  $R_1$  and  $R_2 \cap F_D = \partial R_2 = \partial(D_{p-1} \cup D'_2)$ . Repeat this operation for every pair of a positive disk

$D_{p-i+1}$  and a negative disk  $D'_i$ ,  $1 \leq i \leq q$ , in the same manner, so that we have a sequence of cylinders. Positive disks and bands corresponding to the subgraph  $OB'$  are untouched as in Case 1. This untouched part of  $F_D$  and the cylinders and all bands form our new surface  $F_1$ . See Figure 7.2 (b). For the second case (b2),  $F_1$  is constructed in the same manner, and it looks like a surface depicted in Figure 7.2 (c). We note that in this construction, all bands are untouched, and therefore, we are only concerned with disks.

Case 3:  $\nu(G) \geq 4$ .

Subcase (i): The origin  $O$  is local minimal.

Then a local maximal vertex  $A$  is followed by a local minimal vertex  $B$ . See Figure 7.3 (a1) and (a2).

Subcase (ii): The origin  $O$  is local maximal.

Then a local minimal vertex  $A$  is followed by a local maximal vertex  $B$ . See Figure 7.3 (b1) and (b2).

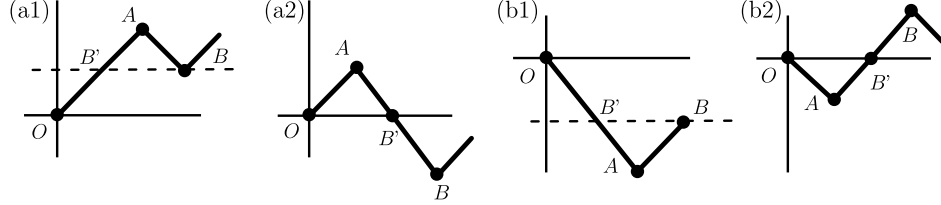


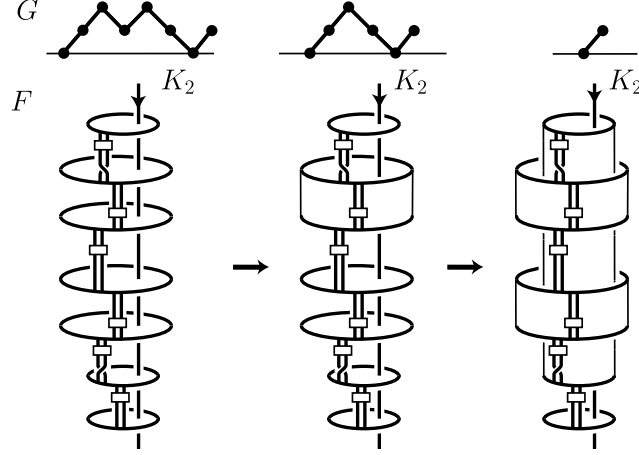
Figure 7.3: Graph  $G$  for  $\nu(G) \geq 4$ , where  $O$  is a local minimal, or maximal

In, Subcase (i) (a1), first, apply the argument used in Case 2 (b1) on the subgraph  $OA \cup AB$  of  $G(S)$ , and replace pairs of disks corresponding to the edges on  $B'A$  and  $AB$  by cylinders. Secondly, delete the subgraph  $B'A \cup AB$  from  $G$  and then identify  $B'$  and  $B$  to obtain a new graph  $G'$ . Since  $\nu(G') = \nu(G) - 2$ , we can inductively construct a surface  $F'_1$  in such a way that all cylinders in  $F'_1$  are inside the cylinders we previously constructed. Our new surface  $F_1$  is the union of cylinders firstly constructed and  $F'_1$ , (and all bands).

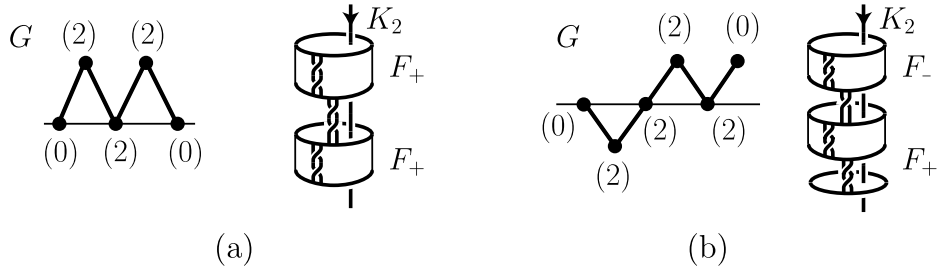
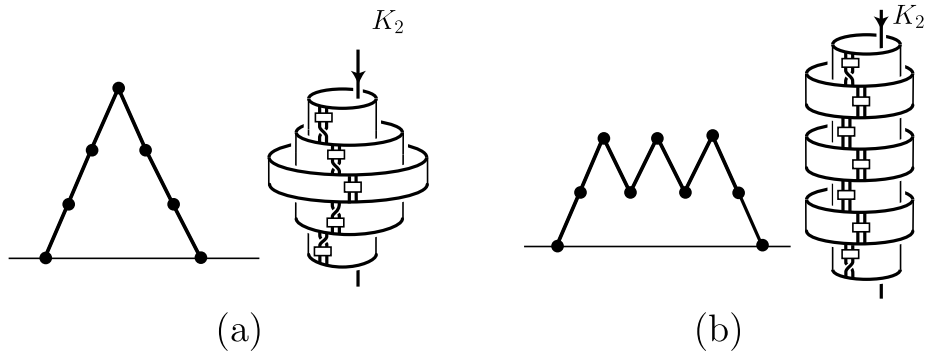
As an example, in Figure 7.4 below, we depict a sequence of modifications of  $G$  and corresponding surfaces. The last surface is the surface  $F_1$  we sought.

In Subcase (i) (a2), apply the argument used in Case 2 (b1) on the subgraph  $OA \cup AB'$ , and replace pairs of disks by cylinders. Then delete the subgraph  $OA \cup AB'$  from  $G(S)$  and identify  $O$  and  $B'$  so that a new graph  $G'$  is obtained. Since  $\nu(G') = \nu(G) - 1$ , apply induction on  $G'$ .

In Subcase (ii) (b1) and (b2), apply the argument used in Case 2 (b2) and repeat similar arguments used in Subcase (i) (a1) and (a2).

Figure 7.4: Construction of a canonical surface from a graph  $G$  (I)

**Example 7.1.** Figures 7.5 and 7.6 depict graphs and corresponding canonical surfaces.

Figure 7.5: Construction of a canonical surface from  $G$  (II)Figure 7.6: Construction of a canonical surface from  $G$  (III)

To construct  $F_1$ , we start with a primitive spanning disk and the genus increases by 1 each time we replace a pair of small disks by a cylinder. We have  $(\lambda - \ell)/2$  pairs of disks to replace. Therefore, we have the following:

**Proposition 7.2.** *Let  $\lambda$  be the number of edges in  $G(S)$ . The number of disks in  $F_1$  constructed above is  $\ell = \ell k B(2\alpha, \beta)$ , and the number of cylinders of  $F_1$  is  $\frac{1}{2}(\lambda - \ell)$ , therefore,  $g(F_1) = \frac{1}{2}(\lambda - \ell)$ .*

Now we twist  $F_1$  by  $K_2$ . First, suppose  $\ell = 0$ . Then just by twisting, we obtain a Seifert surface  $F(r)$  for  $K(r)$ . In Section 10, (10.1), we show that  $g(K(r)) = g(F(r))$ , which is equal to the number of cylinders, i.e.,  $(\lambda - \ell)/2$ .

Next, suppose  $\ell \neq 0$ .  $F_1$  consists of  $\ell$  disks and  $(\lambda - \ell)/2$  cylinders, connected by bands, and  $g(F_1) = \frac{1}{2}(\lambda - \ell)$  (by Proposition 7.2). If we twist  $F_1$ ,  $r$  times by  $K_2$ , we obtain a singular surface, in which cylinders penetrate the  $\ell$  disks transversely. Remove ribbon singularities by smoothing intersections in the standard way. Then we obtain a Seifert surface  $F(r)$  for  $K(r)$ . See Figure 7.7 for the case  $r = 1$ .

For a technical reason, we need more specific description (given at the end of this section) on the position of bands connecting disks and cylinders.

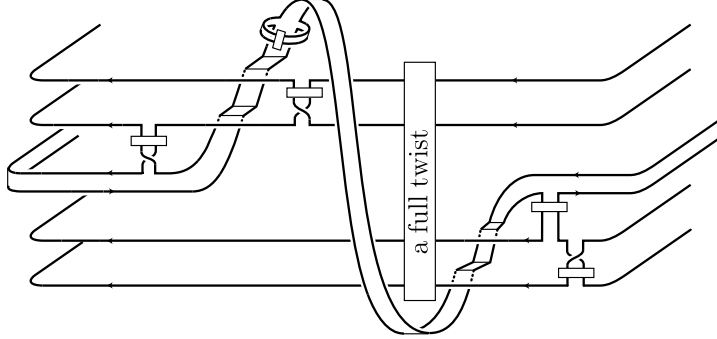


Figure 7.7: A canonical surface  $F(r)$ , where  $r = 1$

Each time we make a hole, the genus of the surface is increased by one. We see that there are exactly  $\frac{1}{2}(\lambda - \ell)\ell r$  intersections. Furthermore, by twisting along  $K_2$ , the boundaries of  $\ell$  disks form a torus link of type  $(\ell, \ell r)$ .

Therefore, we have:

$$\begin{aligned} g(F(r)) &= \frac{1}{2}(\lambda - \ell) + \frac{1}{2}(\lambda - \ell)\ell r + \frac{1}{2}(\ell - 1)\ell r \\ &= \frac{1}{2}\{(\lambda - 1)(\ell r + 1) - (\ell - 1)\}. \end{aligned} \quad (7.1)$$

In the proof of Theorem 2.1, in the next section, we show (i) if  $\rho = 0$ , then  $F(r)$  is a minimal genus Seifert surface for  $K(r)$ , and (ii) if  $\rho > 0$ , then  $F(r)$  admits compressions  $\rho$  times and the result is a minimal genus Seifert surface for  $K(r)$ , where  $\rho$  is the deficiency (Definition 3.10).

Now we give a precise description of the relative position of bands connecting disks and cylinders. Proposition 3.20 also implies that, in  $F_1$ , we have a freedom of relative positions of the bands. However this freedom is lost when we have twisted  $F_1$  to obtain  $F(r)$ .

A band  $B$  in  $F_1$  is of one of the five types below. See Figure 7.8. Note that the

boundary of  $(F_1 - \text{bands})$  consists of circles. In Figure 7.8, a circle with an arrow heading toward left (resp. right) corresponds to a rising (resp. falling) edge of  $G$ . Each band corresponds to a vertex in  $G$ .

Type I:  $B$  connects the boundary of an outermost cylinder.

Type II:  $B$  connects two stacked cylinders.

Type III:  $B$  connects two cylinders side by side (of the same sign or the opposite sign).

Type IV:  $B$  connects a disk and a cylinder. In this type, we have three subtypes as in Figure 7.8.3.

Type V:  $B$  connects two positive disks.

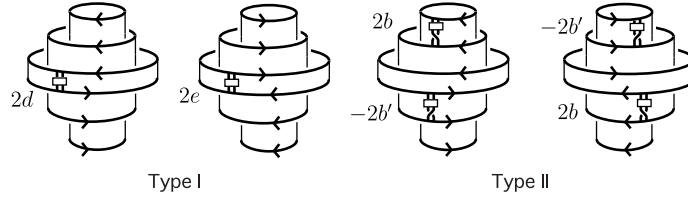


Figure 7.8.1: Bands of Types I and II

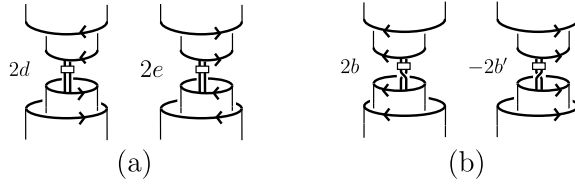


Figure 7.8.2: Bands of Type III

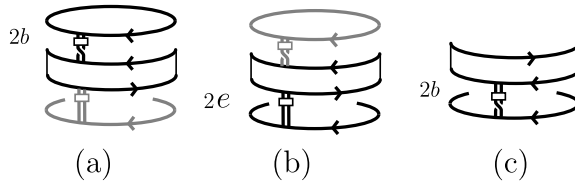


Figure 7.8.3: Bands of Type IV

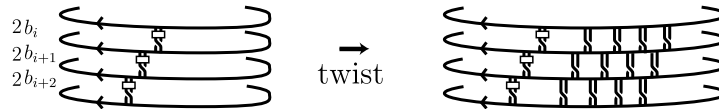


Figure 7.8.4: Bands of Type V

Note. In Figure 7.8.3 (c), there are no disks above the depicted cylinder by our construction of  $F_1$ .

To obtain  $F(r)$ , we place the bands of Type V as in Figure 7.8.4, so that they are placed close to the bands emerged by twisting. The bands of Types I, II and III should be arranged as in Figure 7.9, where the horizontal disk is the very top disk in  $F_1$ , i.e, it corresponds to the edge in  $G$  that is the first rising edge after the last intersection of  $G$  and the  $x$ -axis. Bands of Type IV, where a positive cylinder

is connected (Figure 7.8.3 (a), (b)), should be arranged as in Figure 7.7. If  $r \geq 2$ , then the first band (resp. the second band, if any) is placed before (resp. after) the  $r$ -twists. A band of Type IV, where a negative cylinder is connected (Figure 7.8.3 (c)) is similarly done as Figure 7.8.3 (b). See Figure 8.5 for a local picture for this type of band after twisting by  $K_2$ .

Now we have constructed a Seifert surface for  $K(r)$ .

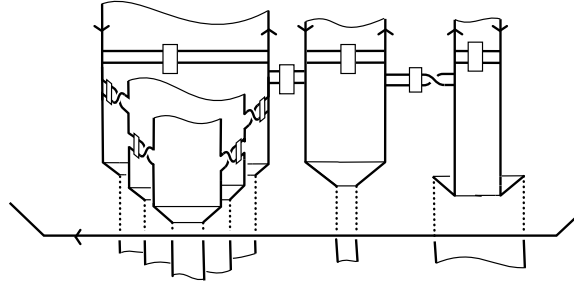


Figure 7.9: Placement of the bands of Types I, II and III

## 8. Proof of Theorem 2.1

Let  $S = \{P_1, d_1, Q_1, e_1, P_2, d_2, Q_2, e_2, \dots, P_m, d_m, Q_m\}$  be the canonical decomposition of the continued fraction of  $\beta/2\alpha$ . Express each  $P_i$  and  $Q_j$  by modified continued fractions, and write,

$$P_i = [[1, b_{i,1}, 1, b_{i,2}, 1, \dots, 1, b_{i,s_i}, 1]], 1 \leq i \leq m,$$

$$Q_j = [[-1, -b'_{j,1}, -1, -b'_{j,2}, -1, \dots, -1, -b'_{j,q_j}, -1]], 1 \leq j \leq m,$$

where  $b_{i,k}$  and  $b'_{j,k}$  are arbitrary, and may be 0. Since  $\ell \neq 0$ , we see from Proposition 4.1,

$$\Delta_{K(r)}(t) = \frac{1-t}{1-t^\ell} \Delta_{B(2\alpha, \beta)}(t, t^{\ell r}), \quad (8.1)$$

where  $\ell > 0$  and  $r > 0$ . Then, by Proposition 6.3, we have:

$$\deg \Delta_{K(r)}(t) = (\lambda - 1)(\ell r + 1) - (\ell - 1) - 2\rho.$$

Recall that in Section 7, (7.1), we constructed a Seifert surface  $F(r)$  for  $K(r)$  with  $g(F(r)) = \frac{1}{2} \{(\lambda - 1)(\ell r + 1) - (\ell - 1)\}$ .

If  $\rho = 0$ , then  $F(r)$  is a minimal genus Seifert surface for  $K(r)$ , since  $g(F(r)) = \frac{1}{2} \deg \Delta_{K(r)}(t)$ . Therefore, to prove Theorem 2.1, it suffices to confirm that we can compress  $F(r)$  as many times as  $\rho$  (Definition 3.10). In Proposition 8.1 below, we demonstrate where to apply compression corresponding to each  $b_{i,k} = 1$  and  $b'_{j,k} = 1$  in  $S$ . Since we need it in the proof of Theorem 2.2, we also show where we can deplumb twisted annuli from  $F(r)$ . But first, we apply compressions.

**Proposition 8.1.** *We can compress  $F(r)$   $\rho$  times, where each compression corresponds to an occasion of  $b_{i,k} = 1$  or  $b'_{j,k} = 1$ . Furthermore, we can deplumb*



$\sum_{i=1}^m s_i + \sum_{j=1}^m q_j + 2m - 1 - \rho$  unknotted, twisted annuli from  $F(r)$ , where each deplumbing corresponds to an occasion of  $b_{i,k} \neq 1, b'_{j,k} \neq 1, d_i$  and  $e_i$ .

*Proof.* At each band connecting disks and cylinders, we explicitly show how to apply either compression or deplumbing of an annulus. Each deplumbing corresponds to removing a band, and each compression corresponds to cutting the surface along a properly embedded arc.

If a band is of Type I, it is obvious that we can deplumb an unknotted annulus with  $d_i$  or  $e_j$  full twists. Compressions never occur for this type. For a band of Type II, the relevant part of  $F(r)$  is depicted in Figure 8.1. If  $b \neq 1$  or  $b' \neq 1$ , then we can deplumb a twisted annulus and thus remove a band. See Figure 8.1 (a). In particular, if  $b$  or  $b'$  is either 0 or 2, then the annulus is a Hopf band. However, if  $b = 1$  or  $b' = 1$ , then the annulus yields a compressing disk, so we do not deplumb a band, but apply compression (see Figure 8.1 (b)).

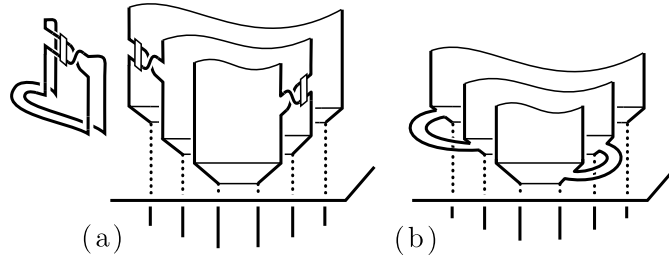


Figure 8.1: Deplumbing and compression for Type II

Take a band  $B$  of Type III. If  $B$  connects two cylinders showing the same side (see Figure 8.2), we can deplumb a twisted annulus. In this case, compression never occurs. In particular we can deplume a Hopf band if  $d = \pm 1$  or  $e = \pm 1$ . On the other hand, if  $B$  connects two cylinders showing the opposite sides (see Figure 8.3), then we can apply compression if  $b$  or  $b'$  equals 1, and otherwise deplumb a twisted annulus.

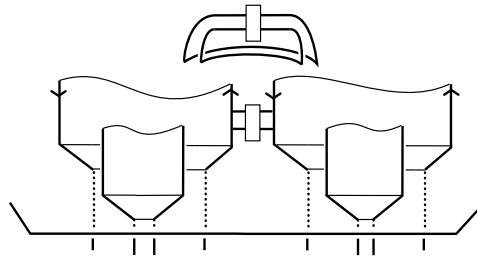


Figure 8.2: Deplumbing an annulus for Type III (a)

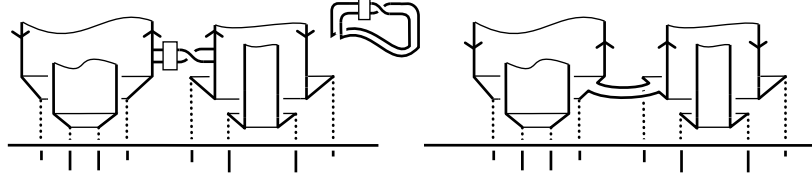


Figure 8.3: Deplumbing and compression for Type III (b)

Take a band  $B$  of Type IV (Figure 7.8.3). There are three subtypes according to the feature of the corresponding vertex  $v_B$  in  $G$ : (a)  $v_B$ , not on the  $x$ -axis is between two rising edges of  $G$ , (b)  $v_B$  is a local minimum, and (c)  $v_B$ , on the  $x$ -axis, is between two rising edges.

For (a), see Figure 8.4. If  $b = 1$ , then we can compress, and otherwise we can remove the band by deplumbing an annulus with  $b - 1$  full twists. (See Figure 7.7.) For (b), we can deplumb a band with  $e$  full twists, and compression never occurs. For (c), see Figure 8.5.

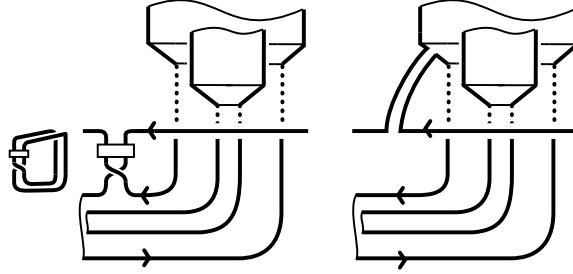


Figure 8.4: Deplumbing and compression for Type IV (a)

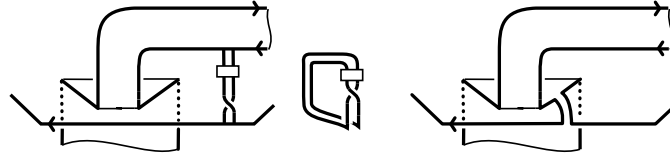


Figure 8.5: Deplumbing and compression for Type IV (c)

For bands of Type V, deform  $F(r)$  by isotopy as in Figure 8.6 so that each band of Type V is adjacent to a band emerged by twisting  $F_1$ . Then we can remove the band by deplumbing if  $b \neq 1$ , and otherwise cancel it with its neighbor, which corresponds to compressing  $F(r)$ . Note that since  $r \geq 1$ , even if we cancel all bands of Type V, still there are bands connecting each pair of adjacent disks.  $\square$

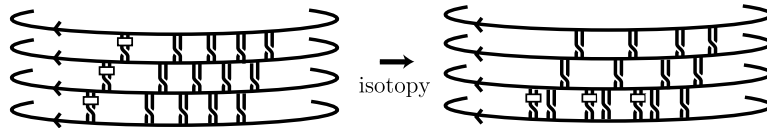


Figure 8.6: Deplumbing and compression for Type V

By Proposition 8.1, the proof of Theorem 2.1 is now completed.

## 9. Proof of Theorem 2.2

By Theorem 3.14, we assume,  $\ell k B(2\alpha, \beta) = \ell \geq 0$  and  $r > 0$ .

Let  $\beta/2\alpha = [[c_1, c_2, \dots, c_{2d+1}]]$  be a continued fraction of  $\beta/2\alpha$ , and  $S = \{P_1, d_1, Q_1, e_1, P_2, \dots, P_m, d_m, Q_m\}$  be the canonical decomposition of  $S$ . Write

$$P_i = [[a_{i,1}, b_{i,1}, a_{i,2}, b_{i,2}, \dots, a_{i,s_i}, b_{i,s_i}, a_{i,s_i+1}]], a_{i,j} > 0, \text{ and} \\ Q_j = [[-a'_{j,1}, -b'_{j,1}, \dots, -a'_{j,q_j}, -b'_{j,q_j}, -a'_{j,q_j+1}]], a'_{j,k} > 0.$$

### 9.1. Reformation of Theorem 2.2.

In Theorem 6.1, we have characterized  $K(2\alpha, \beta|r)$  with  $\ell > 0, r > 0$  whose Alexander polynomial is monic. Hence now Theorem 2.2 is equivalent to Theorem 9.1 below.

**Theorem 9.1.** *Fibredness of  $K(r) = K(2\alpha, \beta|r)$  with  $\ell = \ell k B(2\alpha, \beta) \neq 0$  is determined as follows, where we assume  $\ell > 0$  and  $r > 0$  by Theorem 3.14. In each case below,  $a_{i,j}$  and  $a'_{i,j}$  are arbitrary.*

*Case 1.  $\ell = r = 1$ .  $K(1)$  is fibred if and only if*

- (a)  $d_i, e_j = \pm 1$ , for any  $i, j$ , and
- (b) in each  $P_i$  and  $Q_i$ ,  $b_{i,k}$  and  $b'_{j,p}$  are 2.

*Case 2.  $\ell r \geq 2$ .  $K(r)$  is fibred if and only if*

- (a)  $d_i, e_j = \pm 1$ , for any  $i, j$ , and
- (b) in each  $P_i$  and  $Q_i$ ,  $b_{i,k}$  and  $b'_{j,p}$  are 1 or 2.

### 9.2. Proof of Theorem 9.1, Case 1.

We assume in this subsection

$$\ell = \ell k B(2\alpha, \beta) = 1 \text{ and } r = 1. \quad (9.1)$$

First suppose  $K(r)$  is fibred. Then  $\Delta_{K(r)}(t)$  is monic, and hence, by Proposition 6.5,

$$(a) \ d_i, e_j = \pm 1, \text{ and} \\ (b) \ b_{i,k} = b'_{j,p} = 2 \text{ for any } i, j, k, p. \quad (9.2)$$

This proves the “only if” part of Case 1.

Conversely, suppose (9.2) is satisfied. Rewrite the continued fraction as the modified continued fraction. Then some of new  $b_{i,k}, b'_{j,p}$  may be zero, but still (9.2) implies the deficiency  $\rho = 0$ . Now, by Proposition 8.1, we see that (9.2) also implies the following: Let  $F^*$  be the surface obtained from  $F(r)$  (constructed in Section 7) by removing all the bands connecting the disks and cylinders. Then  $F^*$  is obtained from  $F(r)$  by deplumbing Hopf bands. Therefore, to prove that  $K(r)$  is fibred, it suffices to show that  $F^*$  is a fibre surface.

The following lemma shows that  $F^*$  is a fibre surface, and hence Theorem 2.2 Case 1 is proved. □

**Lemma 9.2.** *(Braided fibre surface) Let  $L_1$  and  $L_2$  be (naturally) oriented closed braids in a tubular neighbourhood  $N(L)$  of a Hopf link  $L$ , where  $L_1$  and  $L_2$  are*

embedded in different components of  $N(L)$ . Suppose that  $L_1$  is a positive closed braid. Then  $L_1$  is a fibred link and a fibre surface  $S$  for  $L_1$  is obtained by applying Seifert algorithm. Now replace each component  $L_{2,i} (i \leq \mu)$  of  $L_2$  by an annulus  $B_i$  whose core is  $K_{2,i}$ , but the number of twists of  $B_i$  is arbitrary. We assume that  $B_i$  intersects  $S$  transversally in ribbon singularities. By smoothing all the ribbon singularities, we obtain a Seifert surface  $F$  for  $L_1 \cup \partial B_1 \cup \cdots \cup \partial B_\mu$ . Then  $F$  is a fibre surface.

*Proof.* The surface  $S$  consists of disks and bands connecting these disks. Since  $L_1$  is a positive braid, each band has only a positive half twist. Since neighbouring two bands form a Hopf band, we can eliminate one of the bands by deplumbing. After all, we may assume that  $L_1$  is a trivial knot, i.e.  $S$  consists of  $\nu$  disks and  $\nu - 1$  bands, and it suffices to show that  $F$  constructed using this  $S$  is a fibre surface. See Figure 9.1.

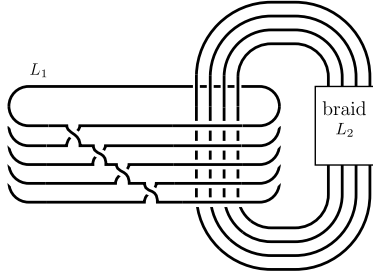


Figure 9.1: A braid penetrating a disk

Denote by  $\mu$  the number of strings of the braid of  $L_2$ . Now consider a sutured manifold  $M = F \times I$ . For the definition of sutured manifold and its decompositions, see [5, pp.8–10 and Appendix A] and [6, Section 1].  $M$  is a solid ball with  $\nu \times \mu$  holes and  $\nu \times \mu$  1-handles attached. Applying a series of  $C$ -product decompositions, first we fill these holes by 2-handles, and obtain a ball  $M'$  with  $\nu \times \mu$  1-handles attached, where the suture on  $M'$  is the equator, and each 1-handle has exactly one suture, which is parallel to a co-core. Since each of the 1-handles connects the north hemisphere and the south hemisphere of  $M'$  without a local knotting, we can arrange the 1-handles by sliding their feet so that they are attached to  $M'$  trivially. Then, by a  $C$ -product decomposition, we can amalgamate a pair of 1-handles, and eventually, we have a solid torus whose sutures are two meridians. Applying one more  $C$ -product decomposition, we have a ball with a single suture. Therefore, the original surface  $F$  is a fibre surface.  $\square$

### 9.3. Proof of Theorem 9.1 Case 2.

In this subsection, we assume that

$$\ell r \geq 2. \quad (9.3)$$

First, we note that Proposition 6.4 proves the “only if” part. Therefore, suppose

that the continued fraction of  $\beta/2\alpha$  satisfies

$$\begin{aligned} (a) & d_i, e_j = \pm 1 \text{ and} \\ (b) & b_{i,k} \text{ and } b'_{j,p} = 1 \text{ or } 2 \text{ for any } i, j, k, p. \end{aligned} \quad (9.4)$$

Again rewrite the continued fraction as a modified continued fraction. Then in (9.4) (b) we have ‘0, 1 or 2’, in stead of ‘1 or 2’.

By Proposition 8.1, we can apply compressions corresponding to all  $b_{i,k}, b'_{j,p} = 1$ , and remove all bands with  $d_i, e_j = \pm 1$  and  $b_{i,k} = b'_{j,p} = 0$  or 2 by deplumbing Hopf bands. Denote the resulting surface by  $\tilde{F}$ . To complete the proof, it suffices to show  $\tilde{F}$  is a fibre surface. Recall that in the proof of Proposition 8.1, each compression corresponds to cutting the surface along a properly embedded arc.

In the following, we depict how to undo each of the cuts by plumbing a Hopf band. To do this, the assumption that  $\ell r \geq 2$  is essential. In fact, to undo the cuts, we use two consecutive holes that occur as intersections of a cylinder and the disk(s).

It suffices to consider each of Types II, III and IV in Proposition 8.1.

For Type II (resp. III), we plumb a band  $B$  along the curve depicted in Figure 9.2 (resp. 9.3). Then we can undo the cut by plumbing a Hopf band.

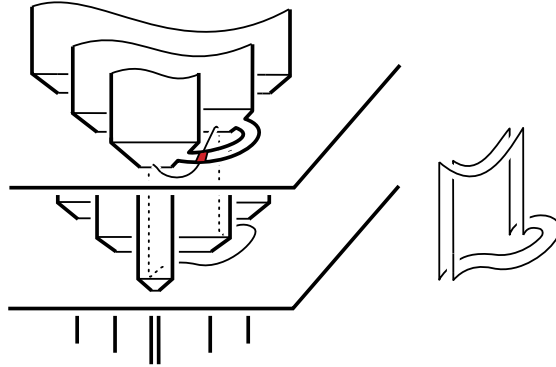


Figure 9.2: Plumbing a Hopf band for Type II

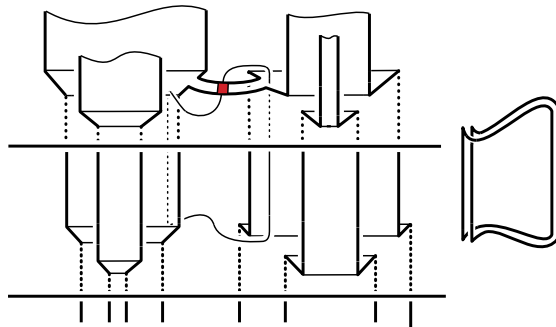


Figure 9.3: Plumbing a Hopf band for Type III

Type IV is a bit complicated. Since the argument is similar, we only prove for the case where the band connects the positive disk and the positive cylinder (in Figure 7.8.3 (a)). The cut made by compression is in Figure 8.4. In Figure 9.4, the band  $B'$  gives a compressing disk so that the result of compression is as in Figure 8.4.

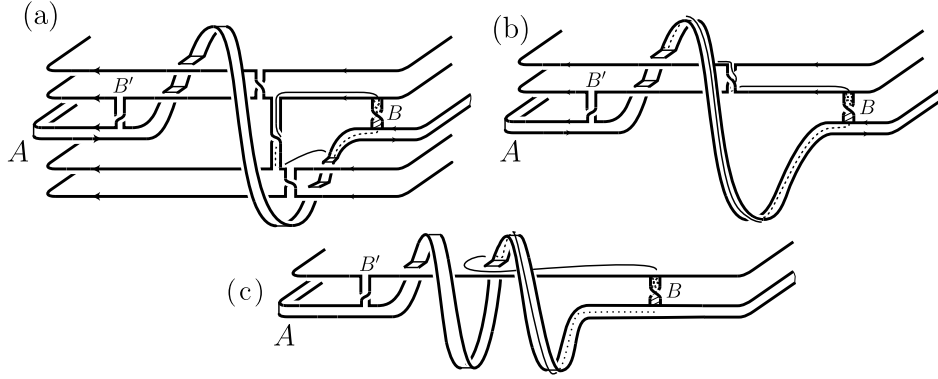


Figure 9.4: Plumbing a Hopf band for Type IV

To undo the cut, we consider three subcases: Let  $A$  be the annulus to which  $B'$  is connected. Subcase (i): There are some disks below  $A$  in  $\tilde{F}$  (Figure 9.4 (a)). Subcase (ii): There are more than one disk above  $A$  but no disks below  $A$  in  $\tilde{F}$  (Figure 9.4 (b)), Subcase (iii), There are only one disk above  $A$  and no disks below  $A$  in  $\tilde{F}$  (Figure 9.4 (c)). Note that in Subcase (iii),  $r \geq 2$  by assumption that  $\ell r > 1$ .

In each subcase, using the arc depicted in Figure 9.4, we can add a band  $B$  by plumbing a Hopf band after compressing at  $B'$ . Then, by sliding  $B$  along  $A$  to the cite of compression, we can undo the cut. This fact is also understood by seeing that the bands  $B$  and  $B'$  cancel each other.

Now all cuts are undone by Hopf plumbings. Then as in Case 1, we can further deplumb Hopf bands and apply Lemma 9.2. This proves Theorem 9.1, Case 2.  $\square$

## 10. Proof of Theorem 2.3

In this section, we determine the genus of a knot  $K(r)$  for the case  $\ell k B(2\alpha, \beta) = \ell = 0$ , and thus prove Theorem 2.3.

In Section 7, we span  $K_1$  by a canonical Seifert surface  $F_1$ . By applying Dehn twists on  $F_1$  along  $K_2$ , we obtain a Seifert surface  $F(r) = F$  for  $K(r)$ . We will show that  $F$  is of minimal genus. Since  $F$  and  $F_1$  have the same genus, we show in fact that  $g(K(r))$  is equal to the number of cylinders in  $F_1$ . However,  $g(F)$  is much larger than one half of the degree of the Alexander polynomial of  $K(r)$ , c.f., Proposition 6.8. Therefore, in order to show that  $F$  is of minimal genus, we use geometry. First, deplumbing a twisted annulus from  $F$  does not hurt the genus-minimality by the additivity of genus under the Murasugi sum. So we remove all bands connecting the boundaries of the same cylinder. Then our main tool is the sutured manifold hierarchies. As a special case of general results of sutured manifold hierarchies, we

have the following (see [5, Corollary 1.29]):

**Proposition 10.1.** *Let  $(M, \gamma) = (F \times I, \partial F \times I)$  be the sutured manifold obtained from a Seifert surface  $F$ . Apply complementary disk- (annulus-) decompositions to  $(M, \gamma)$  and suppose we obtain  $(V, \delta)$  where  $V$  is a standard solid torus and each suture is a loop running longitudinally once and meridionally non-zero times. Then  $F$  is of minimal genus.*

Throughout the rest of this section, we omit the adjective ‘complementary’ for complementary sutured manifold decompositions, since we only deal with such decompositions and no confusions are expected.

Let  $[2u_1, 2v_1, 2u_2, 2v_2, \dots, 2u_m, 2v_m, 2u_{m+1}]$  be the continued fraction of  $\beta/2\alpha$ . Suppose that  $\ell k B(2\alpha, \beta) = 0$ . Then to prove Theorem 2.3, we show the following:

$$g(K(r)) = \frac{1}{2} \sum_{i=1}^{m+1} |u_i| = \frac{\lambda}{2} (= \#\{\text{cylinders of } F_1\}). \quad (10.1)$$

Note that since  $\ell k B(2\alpha, \beta) = 0$ ,  $\sum_{i=1}^{m+1} |u_i| = \lambda = \#\{\text{edges of } G(S)\}$  is even.

*Proof.* We prove (10.1) by induction on  $\lambda$ . If  $\lambda = 2$ , then (10.1) is obvious since  $F(r)$  is a plumbing of two twisted annuli. Suppose  $\lambda \geq 4$ . First we deplumb all bands corresponding to proper local maximal, or minimal vertices, i.e., those connecting the two boundaries of a cylinder. Denote by  $\widehat{F}$  the resulting surface. We inductively reduce the graph  $G(S)$  and accordingly amalgamate the solid tori in  $(\widehat{F} \times I, \partial \widehat{F} \times I)$  by disk- (annulus-) decompositions, until we have only one torus where each of the sutures run longitudinally once and meridionally non-zero times. After that, we will see that all such deplumbing and amalgamations commute with Dehn twists along  $K_2$ , and hence by Proposition 10.1, we have (10.1).

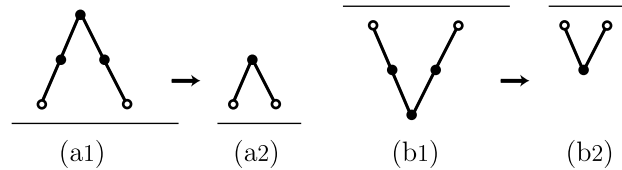


Figure 10.1: Reduction on the graph (I)

Case 1: There is a vertex in  $G(S)$  incident to two consecutive rising edges and two consecutive falling edges, as in Figure 10.1 (a1) or (b1), where the white vertices may be a terminal vertex or a non-terminal vertex, and they may lie on the  $x$ -axes.

As in Figure 10.2, we amalgamate the top solid torus with the second top one. There are several cases according to the number of twists in the two bands connected.

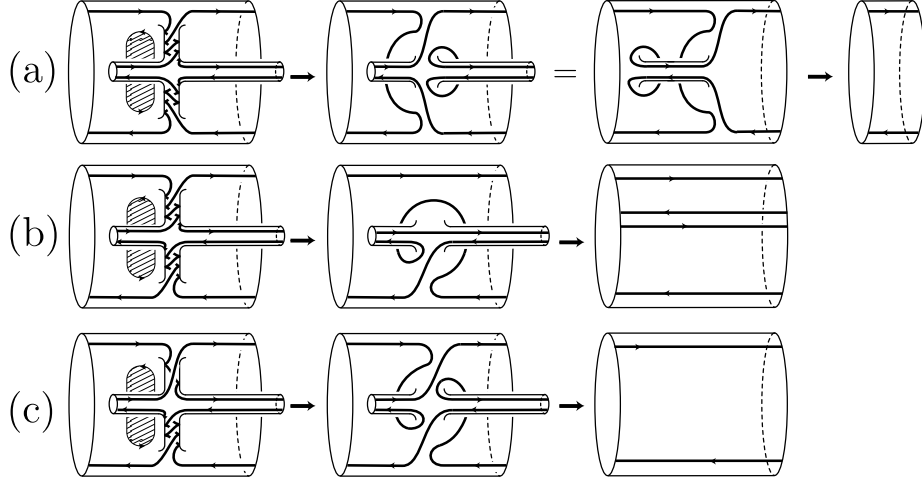


Figure 10.2: Reduction by complementary disk decompositions

Case 1.1. The two bands are twisted in the opposite directions: See Figure 10.2 (a). First apply a disk decomposition using the disk with shadow, then apply a product disk decomposition. Note that if the two bands are both only half-twisted, then the first disk decomposition is also a product disk decomposition.

Case 1.2. Two bands are twisted in the same direction: See Figure 10.2 (b) and (c). As before, we apply a disk decomposition and a product disk decomposition. Figure 10.2 (b) depicts the case where both bands are more than half-twisted. In this case, we have two extra sutures, but they do not affect the following inductive amalgamations. Figure 10.2 (c) depicts the case at least one of the bands is half-twisted. Note that if both bands are half-twisted, then the first disk decomposition is a product disk decomposition.

Case 2: There are no subgraph considered in Case 1, or all such subgraphs are removed. Now it suffices to find a subgraph as in Figure 10.3 or 10.4 and amalgamate a solid torus.

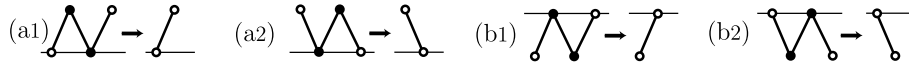


Figure 10.3: Reduction on the graph (II)

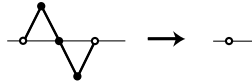


Figure 10.4: Reduction on the graph (III)

Since the other cases are similar, we only consider subgraphs in Figure 10.3 (a1) and Figure 10.4.

By construction, a rising edge above (resp. below) the  $x$ -axes is paired with the a falling edge on its right (resp. left) so that the pair corresponds to a cylinder in  $F_1$ . Sutured manifold decompositions amalgamate the solid tori as respectively depicted in Figures 10.5 and 10.6. In Figure 10.5, corresponding to Figure 10.3 (a1),



two cylinders showing the same side are connected by an even-twisted band. In Figure 10.6, corresponding to Figure 10.4, two cylinders showing the opposite sides are connected by an odd-twisted band. However, as seen in Figure 10.2, longitudinal sutures may have accumulated. In Figures 10.5 and 10.6, we first apply an annulus decomposition and then a disk decomposition. Now we have amalgamated all the tori into one and see that we may apply the Dehn twist along  $K_2$  beforehand. Therefore, by Proposition 10.1, (10.1) is obtained.  $\square$

The proof of Theorem 2.3 is now completed.

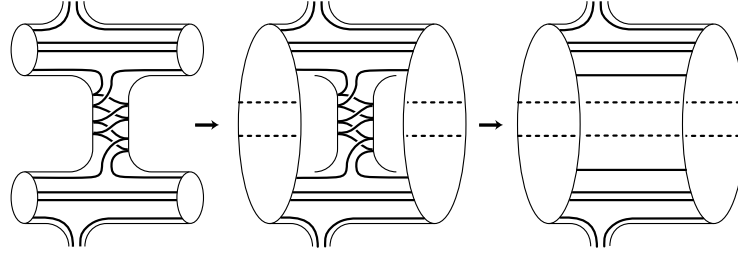


Figure 10.5: Amalgamation of tori (I)

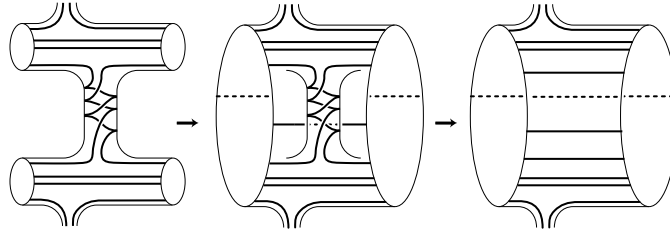


Figure 10.6: Amalgamation of tori (II)

As an immediate consequence of Theorem 2.3, we have:

**Corollary 10.2.** *Suppose  $\ell k B(2\alpha, \beta) = 0$  and  $\alpha \geq 2$ . Then, for any  $r > 0$ ,  $K(2\alpha, \beta|r)$  is never unknotted.*

## 11. Proof of Theorem 2.4

*Proof of Theorem 2.4 (a).* Write  $\Delta_{B(2\alpha, \beta)}(x, y) = (x - 1)(y - 1)f(x, y)$ . Then by Proposition 4.1, we can write  $\Delta_{K(r)}(t) = r(t - 1)^2 f(t, 1) + \varepsilon t^k$ , where  $\varepsilon = \pm 1$  and  $k$  is chosen so that  $\Delta_{K(r)}(t)$  is symmetric. Suppose  $r \geq 2$ . If  $f(t, 1) \neq 0$ , then  $\Delta_{K(r)}(t)$  is not monic and hence  $K(r)$  is not fibred. Suppose  $f(t, 1) = 0$ . Then  $\Delta_{K(r)}(t) = 1$ . However, since  $\alpha \geq 2$ ,  $K(2\alpha, \beta|r)$  is not trivial by Corollary 10.2, and therefore  $K(r)$  is not fibred for  $r \geq 2$ .  $\square$

The rest of this section is devoted to the proof of Theorem 2.4 (b). As we remarked in subsection 3.4, we may assume  $r > 0$  and  $\beta > 0$ . To prove Theorem 2.4 (b), we need the following two propositions.

Let  $\{P_1, d_1, Q_1, e_1, P_2, d_2, Q_2, e_2, \dots, P_m, d_m, Q_m\}$  be the canonical decomposition of  $\beta/2\alpha$ . Using Theorem 2.3, first we prove the following.

**Proposition 11.1.**

*Suppose  $\ell k B(2\alpha, \beta) = 0$ . Then for any  $r \geq 1$ ,  $g(K(r)) = \frac{1}{2} \deg \Delta_{K(r)}(t)$  if and only if  $m = 1$ , i.e.  $\{P_1, d_1, Q_1\}$  is the canonical decomposition of  $\beta/2\alpha$ .*

*Proof.* By Theorem 2.3, for any  $r \geq 1$ ,  $2g(K(r)) = \sum_{i=1}^m \left\{ \sum_{k=1}^{s_i+1} |a_{i,k}| + \sum_{k=1}^{q_i+1} |a'_{i,k}| \right\}$ .

On the other hand,  $\deg \Delta_{K(r)}(t) \leq 2\max\{h, |q|\}$ , where  $h$  (and  $q$ ) is the  $y$ -coordinate of the absolute maximal (and minimal) vertices in  $G(S)$  (Proposition 6.8). Therefore, if there exist local minimal vertices (not end vertices), we see easily that  $2g(K(r)) > 2\max\{h, |q|\} \geq \deg \Delta_{K(r)}(t)$ , and hence, there is only one local (and hence, absolute) maximal vertex, and therefore,  $m = 1$ .  $\square$

**Proposition 11.2.** *Suppose  $\ell k B(2\alpha, \beta) = 0$ . Suppose further,  $\{P_1, d_1, Q_1\}$  is the canonical decomposition of  $\beta/2\alpha$ . Then  $\Delta_{K(1)}(t)$  is monic if and only if  $d_1 = \pm 1$ .*

*Proof.* This follows from Corollary 6.9.  $\square$

Now we proceed to the proof of Theorem 2.4(b).

*Proof of the “if” part.* Suppose that the modified continued fraction of  $\beta/2\alpha$  is of the form  $S = [[1, b_1, 1, b_2, \dots, 1, b_{p-1}, 1, d_1, -1, -b'_{p-1}, -1, \dots, -b'_1, -1]]$ , where  $b_i$  and  $b'_i$ ,  $1 \leq i \leq p-1$ , are 0 or 1 and  $d_1 = \pm 1$ . What is to show is that  $K(1) = K(2\alpha, \beta|1)$  is fibred. Let  $F_1$  be the canonical Seifert surface for  $K_1$ . (See Figure 7.6 (a).) By twisting  $F_1$  once by  $K_2$ , we obtain a Seifert surface for  $K$ . Since  $d_1 = \pm 1$ , the band corresponding to the maximal vertex, is a Hopf band, and hence, we may remove it by deplumbing. Denote the resulting surface by  $\hat{F}$ . Now, since  $b_i$  and  $b'_i$  ( $1 \leq i \leq p-1$ ) are either 0 or 1, every band in  $\hat{F}$  is only half-twisted. Therefore, every disk decompositions employed in the proof of (10.1) Case 1 is in fact a product disk decomposition, and hence  $\hat{F}$  is a fibre surface by [6, Theorem 1.9], and  $K(1)$  is a fibred knot.

*Proof of the “only if” part.* Suppose  $K(1) = K(2\alpha, \beta|1)$  is a fibred knot. Then by Proposition 11.1, the continued fraction  $S$  for  $\beta/2\alpha$  must be  $\{P_1, d_1, Q_1\}$ . Therefore, the modified continued fraction is of the form:

$$\beta/2\alpha = [[1, b_1, 1, b_2, \dots, 1, b_{p-1}, 1, d_1, -1, -b'_{p-1}, -1, \dots, -1, -b'_1, -1]].$$

Consider the canonical Seifert surface  $F_1$  for  $K_1$ , as in Figure 7.6 (a). A Seifert surface  $F$  for  $K(1)$ , consisting of  $p$  cylinders  $A_1, A_2, \dots, A_p$  and  $2p-1$  bands, is obtained from  $F_1$  by applying a Dehn twist once along  $K_2$ . In Section 10, we have seen that  $F$  is of minimal genus, and hence a fiber surface for  $K(1)$ . By Proposition 11.2,  $d_1 = \pm 1$  and hence we remove the band on the top annulus  $A_p$  by deplumbing a Hopf band, and denote by  $\hat{F}$  the resulting surface. Then the inclusion map below must be onto:

$$\phi : \pi_1(\hat{F}) \longrightarrow \pi_1(S^3 - \hat{F}). \quad (11.1)$$

Now, we show that all  $b_j$  and  $b'_j$  are 0 or 1 and hence that if at least one of  $b_j$  and  $b'_j$  is neither 0 nor 1, then  $\phi$  is not onto and hence  $K(1)$  is not fibred.

If  $b_1$  and  $b'_1$  are 0 or 1, then we can ‘remove’ the bottom annulus  $A_1$  by two product decompositions as in the proof of (10.1) Case 1. Therefore by [6, Lemma 2.2], we may assume without loss of generality that  $b'_1 \neq 0, 1$ , i.e., the band  $B'_1$  is more than half-twisted. To show that  $\phi$  is not onto, we need explicit presentations of the groups  $\pi_1(\widehat{F})$  and  $\pi_1(S^3 - \widehat{F})$ . To do that, we deform  $\widehat{F}$  as depicted in Figure 11.1 (where  $p = 3$ ). Note that both  $\pi_1(\widehat{F})$  and  $\pi_1(S^3 - \widehat{F})$  are free of rank  $2p - 1$ .

Figure 11.1 also depicts a base point  $**$  and the generators of  $\pi_1(S^3 - \widehat{F})$  denoted by  $x_1, x_2, \dots, x_p$  and  $a_1, a_2, \dots, a_{p-1}$ . Take a base point  $*$  for  $\pi_1(\widehat{F})$  on  $A_1$  as in Figure 11.1. The generators for  $\pi_1(\widehat{F})$  are denoted by  $\alpha_1, \alpha_2, \dots, \alpha_p$  and  $\beta_1, \beta_2, \dots, \beta_{p-1}$ . A loop  $\alpha_i$  starts at  $*$  moving toward on  $A_i$  through bands  $B_1, \dots, B_{i-1}$  and circle once around  $A_i$  counter-clockwise, and then return to  $*$  through  $B_{i-1}, \dots, B_1$ . A loop  $\beta_i$  starts at  $*$  moving toward  $A_{i+1}$  through  $B_1, B_2, \dots, B_i$  and returns to  $*$  passing through first  $B'_i$  and then  $B_{i-1}, B_{i-2}, \dots, B_1$ .

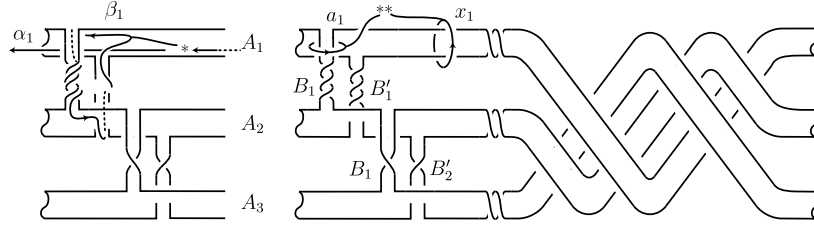


Figure 11.1: Generators of  $\pi_1(S^3 - \widehat{F})$  and  $\pi_1(\widehat{F})$

We must express  $\phi(\alpha_i), \phi(\beta_j)$  in terms of  $x_i, a_j$ . Let  $D = x_1 x_2 \dots x_p$ . For simplicity, we use  $\alpha_i$ , (or  $\beta_j$ ) instead of  $\phi(\alpha_i)$  (or  $\phi(\beta_j)$ ). Then we have the following:

$$\begin{aligned} \alpha_1 &= a_1 D, \\ \alpha_2 &= u_1 a_2 D u_1^{-1}, \\ \alpha_3 &= u_1 u_2 a_3 D u_2^{-1} u_1^{-1}, \\ &\vdots \\ \alpha_{p-1} &= u_1 u_2 \dots u_{p-2} a_{p-1} D u_{p-2}^{-1} \dots u_1^{-1}, \\ \alpha_p &= u_1 u_2 \dots u_{p-1} D u_{p-1}^{-1} \dots u_1^{-1}. \end{aligned} \tag{11.2}$$

$$\begin{aligned} \beta_1 &= u_1 w_1, \\ \beta_2 &= u_1 (u_2 w_2) u_1^{-1}, \\ \beta_3 &= u_1 u_2 (u_3 w_3) u_2^{-1} u_1^{-1}, \\ &\vdots \\ \beta_{p-1} &= u_1 u_2 \dots u_{p-2} (u_{p-1} w_{p-1}) u_{p-2}^{-1} \dots u_1^{-1}, \end{aligned} \tag{11.3}$$

where  $u_i = a_i^{b_i}$  and  $w_i = x_{i+1}^{-1} (x_i^{-1} \dots x_1^{-1} D x_1 \dots x_i D^{-1} a_i^{-1})^{b'_i}$ ,  $1 \leq i \leq p-1$ .

Denote  $h_i = x_i^{-1} \dots x_1^{-1} D x_1 \dots x_i D^{-1} a_i^{-1}$ . By assumption,  $b'_1 \neq 0, 1$ . Let  $H = \text{Im} \phi$  and  $G = \pi_1(S^3 - \widehat{F})$ . We want to show that  $H \neq G$ . First we may suppose that  $a_1 \in H$  and  $x_1 x_2 \in H$ . Otherwise, obviously,  $H \neq G$ , and we are done. Now since  $a_1 \in H$ , by (11.2), we have  $D \in H$ , and hence,  $a_2 \in H$ , since  $H \ni \alpha_2 = u_1 a_2 D u_1^{-1}$ , and  $H \ni u_1 = a_1^{b'_1}$  and  $H \ni D$ . Repeat this process to obtain

$$a_1, a_2, \dots, a_{p-1} \in H \text{ and } D \in H. \quad (11.4)$$

Therefore,  $H$  is generated by

$$\{a_1, a_2, \dots, a_{p-1}, D, x_1 x_2, \alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_{p-1}\}. \quad (11.5)$$

However, since  $\alpha_i$  is written in terms of  $a_i$  and  $D$ , we can eliminate these  $\alpha_i$  from the set of generators (11.5), and hence  $H$  is generated by

$$\{a_1, a_2, \dots, a_{p-1}, D, x_1 x_2, \beta_1, \dots, \beta_{p-1}\}. \quad (11.6)$$

Since  $u_i \in H, 1 \leq i \leq p-1$ , we introduce new generators  $\gamma_j$ , replacing  $\beta_j$ , as

$$\gamma_1 = w_1, \gamma_2 = w_2, \dots, \gamma_{p-1} = w_{p-1}. \quad (11.7)$$

Then  $H$  is generated by

$$\{a_1, a_2, \dots, a_{p-1}, D, x_1 x_2, \gamma_1, \dots, \gamma_{p-1}\}. \quad (11.8)$$

Since  $h_2^{b'_2} = (x_2^{-1} x_1^{-1} D x_1 x_2 D^{-1} a_2^{-1})^{b'_2} \in H$  and  $H \ni \gamma_2 = x_3^{-1} h_2^{b'_2}$ , it follows that  $x_3^{-1} \in H$ . Similarly, using  $\gamma_3, \dots, \gamma_{p-1}$ , we have  $x_4^{-1}, \dots, x_p^{-1} \in H$ . Therefore, we can replace  $x_{i+1}$  by  $\gamma_i, i \geq 2$ , and we see that  $H$  is generated by

$$\{a_1, a_2, \dots, a_{p-1}, D, x_1 x_2, x_3, \dots, x_p, \gamma_1\}. \quad (11.9)$$

Since  $D = x_1 x_2 \dots x_p$ , we can eliminate  $D$  from the set of generators of  $H$ , and  $H$  is generated by  $2p-1$  elements

$$\{a_1, a_2, \dots, a_{p-1}, x_1 x_2, x_3, \dots, x_p, \gamma_1\}. \quad (11.10)$$

Since  $H$  is free of rank  $2p-1$ , the above set is a set of free generators of  $H$ .

On the other hand,  $G$  is freely generated by  $\{a_1, a_2, \dots, a_{p-1}, x_1, x_2, x_3, \dots, x_p\}$ .

Now introduce a new generator  $z_2 = x_1 x_2$  and replace  $x_2$  by  $z_2$ . Then we have:

- (1)  $G$  is generated (freely) by  $\{a_1, a_2, \dots, a_{p-1}, x_1, z_2, x_3, \dots, x_p\}$ , and
- (2)  $H$  is generated (freely) by  $\{a_1, a_2, \dots, a_{p-1}, z_2, x_3, \dots, x_p, \gamma_1\}$ ,

$$\text{where } \gamma_1 = w_1 = x_2^{-1} (x_1^{-1} D x_1 D^{-1} a_1^{-1})^{b'_1}. \quad (11.11)$$

Therefore, if  $H = G$ , then  $x_1 \in H$ . In other words,  $x_1$  can be written as a word on  $a_i, 1 \leq i \leq p-1, z_2, x_j, 3 \leq j \leq p$ , and  $\gamma_1$ . (Note that  $H$  is a free group of rank  $2p-1$ .)

Recall  $b'_1 \neq 0, 1$ .

Case 1:  $b'_1 > 0$ , i.e.  $b'_1 \geq 2$ . Then,

$$\begin{aligned}\gamma_1 &= x_2^{-1}(x_1^{-1}Dx_1D^{-1}a_1^{-1})(x_1^{-1}Dx_1D^{-1}a_1^{-1})^{b'_1-1} \\ &= z_2^{-1}Dx_1D^{-1}a_1^{-1}(x_1^{-1}Dx_1D^{-1}a_1^{-1})^{b'_1-1}.\end{aligned}$$

Since  $z_2^{-1}$ ,  $D$  and  $D^{-1}a_1^{-1}$  are in  $H$ , we can replace  $\gamma_1$  by

$$\delta_1 = x_1(D^{-1}a_1^{-1})(x_1^{-1}Dx_1D^{-1}a_1^{-1})^{b'_1-2}(x_1^{-1}Dx_1).$$

Since  $D = z_2x_3 \cdots x_p$ ,  $D$  does not involve  $x_1$  and hence  $\delta_1$  is of the form:  $\delta_1 = x_1W_1(x_1^{-1}W_2x_1W_1)^{b'_1-2}(x_1^{-1}W_2x_1)$ , where  $W_1 = D^{-1}a_1^{-1}$  and  $W_2 = D$ , none of which involves  $x_1$ . Therefore,  $\delta_1$  is a reduced word. Since  $b'_1 - 2 \geq 0$ ,  $\delta_1$  involves  $x_1$  at least three times, and  $\delta_1$  is of the form:  $x_1Ux_1$ . Therefore, we cannot write  $x_1$  in terms of  $a_i$ ,  $1 \leq i \leq p-1$ ,  $z_2, x_3, \dots, x_p$  and  $\delta_1$ .

Case 2:  $b'_1 < 0$ . Write  $b'_1 = -q$ ,  $q \geq 1$ . Then

$\gamma_1 = x_2^{-1}(a_1Dx_1^{-1}D^{-1}x_1)^q = z_2^{-1}x_1(a_1Dx_1^{-1}D^{-1}x_1)^q$ . Again, since  $z_2 \in H$ , we can replace  $\gamma_1$  by  $\delta'_1$ ,  $\delta'_1 = x_1(a_1Dx_1^{-1}D^{-1}x_1)^q = x_1(W_1^{-1}x_1^{-1}W_2^{-1}x_1)^q$ .

Since  $q > 0$ ,  $\delta'_1$  is a reduced word and  $\delta'_1$  is of the form  $x_1Vx_1$ , and  $V$  contains  $x_1$  at least once. Therefore  $x_1$  cannot be written in terms of  $\delta'_1, a_i, 1 \leq i \leq p-1, z_2, x_3, \dots, x_p$ . It proves that  $H \neq G$ , and hence  $\phi$  is not onto. Theorem 2.4 is now completely proved.  $\square$

## 12. Examples

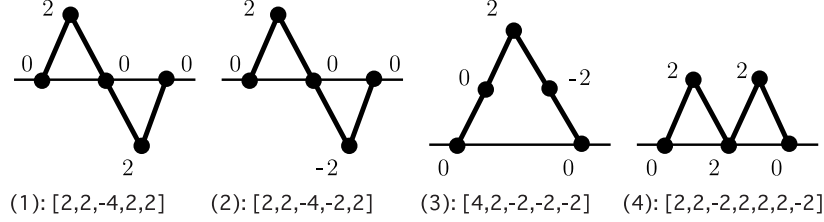
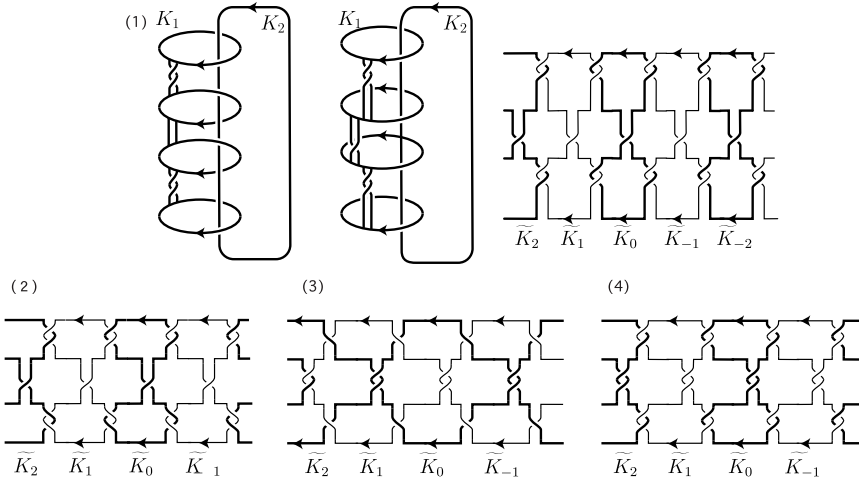
In this section, we discuss several examples that illustrate our main theorems.

**Example 12.1.** All 2-bridge links in this example have linking number 0. Theorem 2.3 determines  $g(K(2\alpha, \beta|r))$  and the fibredness is checked by Theorem 2.4.

(1) Consider  $K(48, 31|r)$ . Since  $31/48 = [2, 2, -4, 2, 2]$ , the genus is 2, and it is not fibred for any  $r > 0$ . The graph is given in Figure 12.1 (1). The lifts  $\{\tilde{K}_j\}$  of  $K_1$  in the infinite cyclic cover  $M^3$  of  $S^3 \setminus K_2$  are depicted in Figure 12.2 (1). From it, we see that  $\Delta_{K(48, 31|1)}(t)$  is not monic. This can also be checked by evaluating  $\Delta_{B(48, 31)}(x, y)$ :

$\Delta_{B(48, 31)}(x, y) = (1-x)(1-y)\{1 - (x+y) - xy(x+y) + x^2y^2\}$ , and hence,  $\Delta_{K(48, 31|r)}(t) = 2r(1-t)^2 + t$ . Thus, we see that  $\Delta_{K(48, 31|1)}(t)$  is not monic.

(2) Consider  $K(64, 41|r)$ . Since  $41/64 = [2, 2, -4, -2, 2]$ , the genus is 2, and it is not fibred for any  $r > 0$ . See Figure 12.1 (2) for its graph. The lifts  $\{\tilde{K}_j\}$  of  $K_1$  in  $M^3$  is depicted in Figure 12.2 (2). From it, we see that  $\Delta_{K(64, 41|r)}(t) = 1$ . The same result is also obtained using  $\Delta_{B(64, 41)}(x, y)$ .  $\Delta_{B(64, 41)}(x, y) = (1-x)^2(1-y)^2(1+xy)$ , and hence  $\Delta_{K(64, 41|r)}(t) = 1$  for any  $r > 0$ .

Figure 12.1: The graphs for 2-bridge links with  $\ell k = 0$ Figure 12.2: Lifts of  $K_1$  in  $M^3$ 

(3) Consider  $K(40, 11|r)$ . Since  $11/40 = [4, 2, -2, -2]$ , the genus is 2 and it is fibred only for  $r = 1$ . See Figure 12.1(3) for its graph. The lifts are depicted in Figure 12.2 (3). Also, we have:  $\Delta_{B(40,11)}(x, y) = (1-x)(1-y)\{(x+y) - xy + xy(x+y)\}$  and hence,  $\Delta_{K(40,11|r)}(t) = r(1-t-t^3+t^4) + t^2$ .

(4) Consider  $K(112, 71|r)$ . Since  $71/112 = [2, 2, -2, 2, 2, -2]$ , the genus is 2 and it is not fibred for any  $r > 0$ . The lifts  $\{\bar{K}_j\}$  of  $K_1$  are depicted in Figure 12.2 (4). Also, we have:  $\Delta_{B(112,71)}(x, y) = (1-x)(1-y)\{1-2(x+y) + (x+y)^2 - 2xy(x+y) + x^2y^2\}$ , and hence,  $\Delta_{K(112,71|r)}(t) = 2r(1-t)^2 + t$ .

**Example 12.2.** Each of the first two 2-bridge links has linking number 1, and the other two links have linking number 2. The graphs are depicted in Figure 12.3. We use Theorem 2.2, Proposition 6.3 and Theorem 6.1.

(1) Consider  $K(18, 13|r)$ . Since  $13/18 = [2, 2, 2, -2, -2]$ , we see  $P_1 = [[1, 1, 1]]$ ,  $Q_1 = [[-1]]$  and  $d_1 = -1$ . Since  $b_{11} = 1$ , it follows from Theorem 6.1 that  $\Delta_{K(18,13|r)}(t)$  is not monic for  $r = 1$ , but it is monic for  $r > 1$ , and hence  $K(18, 13|r)$  is fibred for  $r > 1$ . Since  $\lambda = 3$  and  $\rho = 1$ , the degree of  $\Delta_{K(18,13|r)}(t)$  is  $2r$ , and hence, the genus is  $r$  by Proposition 6.3 and Theorem 2.2. These facts are also confirmed by evaluating  $\Delta_{B(18,13)}(x, y)$ :  $\Delta_{B(18,13)}(x, y) = (x+y) - (x^2 + 3xy + y^2) + xy(x+y)$ , and use Proposition 4.1 (1).

(2) Consider  $K(482, 381|r)$ . Since  $381/482 = [2, 2, 2, 2, -4, -2, -2, 2, 2, 2, 2]$ , we see

that  $P_1 = [[1, 1, 1]]$ ,  $Q_1 = [[-2, -1, -1]]$ ,  $P_2 = [[1, 1, 1]]$ ,  $d_1 = 1$  and  $e_1 = 1$ . Therefore, we see that  $\lambda = 7$  and  $\rho = 3$ , and hence, by Proposition 6.3, the degree of  $\Delta_{K(482,381)}(t) = 6r$  and the genus is  $3r$ . Further, it follows from Theorem 6.1(1) that it is not fibred for  $r = 1$ , but it is fibred for  $r > 1$ . These facts are also confirmed by evaluating the Alexander polynomial of  $B(482, 381)$ , and using Proposition 4.1 (1).  $\Delta_{B(482,381)}(x, y) = (-x^3 + 2x^2 - x)y^6 + (-3x^4 + 8x^3 - 8x^2 + 4x - 1)y^5 + (-3x^5 + 12x^4 - 17x^3 + 14x^2 - 8x + 2)y^4 + (-x^6 + 8x^5 - 17x^4 + 21x^3 - 17x^2 + 8x - 1)y^3 + (2x^6 - 8x^5 + 14x^4 - 17x^3 + 12x^2 - 3x)y^2 + (-x^6 + 4x^5 - 8x^4 + 8x^3 - 3x^2)y - x^5 + 2x^4 - x^3$ . (3) Consider  $K(60, 47|r)$ . Note that  $lkB(60, 47) = 2$ . Since  $47/60 = [2, 2, 2, 2, -2, -2, 2]$ , we see that the genus is  $3r$  and  $\Delta_{K(60,47|r)}(t)$  is monic and hence is fibred for any  $r > 0$ . Note that  $\Delta_{B(60,47)}(x, y) = (x + y) - (2x^2 + 3xy + 2y^2) + (x^3 + 5x^2y + 5xy^2 + y^3) - xy(2x^2 + 3xy + 2y^2) + x^2y^2(x + y)$ .

(4) Consider  $K(1732, -671)$ . Since  $-671/1732 = [-2, 2, 4, 2, -2, 2, 2, 4, 2]$ , we see by Theorem 6.1 (2) that  $\Delta_{K(r)}(t)$  is monic for  $r > 0$  and hence  $K(r)$  is fibred for any  $r > 0$ . Also, since  $\lambda = 6$  and  $\rho = 0$ , we have  $g(K(r)) = 5r + 2$ . Note that  $\Delta_{K(1732,-671)}(x, y) = (x^5 - 5x^4 + 9x^3 - 7x^2 + 2x)y^5 - (5x^5 - 23x^4 + 44x^3 - 42x^2 + 18x - 2)y^4 + (9x^5 - 44x^4 + 87x^3 - 86x^2 + 42x - 7)y^3 - (7x^5 - 42x^4 + 86x^3 - 87x^2 + 44x - 9)y^2 + (2x^5 - 18x^4 + 42x^3 - 44x^2 + 23x - 5)y + 2x^4 - 7x^3 + 9x^2 - 5x + 1$ .

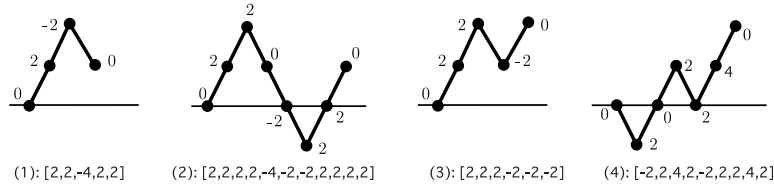


Figure 12.3: More graphs for 2-bridge links with  $lk \neq 0$

### 13. Fibred satellite knots of tunnel number one

In this section, we determine the genera of the satellite knots of tunnel number one, and also solve the question of when it is fibred.

Let  $\widehat{K}$  be a satellite knot of tunnel number one. According to [17], the companion of  $\widehat{K}$  is a torus knot of type  $(a, b)$ , say, and  $|a|, |b| > 1$ , and the pattern is the tori-rational knot  $K(2\alpha, \beta|ab)$  for  $|\alpha| > 1$ . To be more precise, let  $B(2\alpha, \beta) = K_1 \cup K_2$  be a 2-bridge link. Then by construction, a tori-rational knot,  $K(2\alpha, \beta|r)$  is a knot in the interior of a (unknotted) solid torus  $V$ , where  $V$  is homeomorphic to  $S^3 \setminus N(K_2)$ ,  $N(K_2)$  being a tubular neighbourhood of  $K_2$ . Let  $m$  be a meridian of  $V$ . Then the pattern is a pair  $(V, K(2\alpha, \beta|r))$ . Generally, if the pattern is  $(V, K(2\alpha, \beta|r))$ , then our technique can be applied on any satellite knot with fibred companion  $K_C$ . Therefore, in this section we can prove slightly more general theorems as follows.

**Theorem 13.1.** *Let  $K_C$  be a non-trivial fibred knot and  $\widehat{K}$  be the satellite knot with companion  $K_C$  and the pattern  $(V, K(2\alpha, \beta|r))$ ,  $r \neq 0$ . Suppose  $\ell = lk(K_1, K_2) \neq 0$ .*

Then the following hold:

- (1) the genus of  $\widehat{K}$  is exactly half of the degree of the Alexander polynomial of  $\widehat{K}$ .
- (2)  $\widehat{K}$  is fibred if (and only if) the Alexander polynomial of  $\widehat{K}$  (and hence, that of  $K(2\alpha, \beta|r)$ ) is monic.

**Theorem 13.2.** *Under the same notation of Theorem 13.1, suppose  $r \neq 0$  and  $\ell = 0$ . Then (1)  $\widehat{K}$  is not fibred for any  $r \neq 0$ . [3, Proposition 4.15] (2) Let  $[2c_1, 2c_2, \dots, 2c_m]$  be the continued fraction of  $\beta/2\alpha$ . Then  $g(\widehat{K}) = \sum_{\text{odd } j} |c_j|$ .*

*Proof of Theorem 13.1.* (1) Let  $\phi$  be a faithful homeomorphism of a solid torus  $V$  in which  $K(2\alpha, \beta|r)$  is embedded to a tubular neighbourhood  $N(K_C)$  of  $K_C$  in  $S^3$ . The minimal genus Seifert surface  $F$  for  $K = K(2\alpha, \beta|r)$  we had in Section 9 intersects  $\partial V$  at  $\ell$  parallel longitudes, where  $\ell = \ell k(K_1, K_2) > 0$ . Since the image of each longitude under  $\phi$  spans a fiber surface  $S_C$  of genus  $g(K_C)$  in  $S^3 - \phi(V)$ , we can construct a Seifert surface  $\widehat{F}$  for  $\widehat{K}$  by capping off  $\phi(V \cap F)$  by  $\ell$  copies of  $S_C$ . Hence we have  $g(\widehat{K}) \leq g(K) + \ell g(K_C)$ . Combining with Schubert's inequality [3, Proposition 2.10], we have:

$$g(\widehat{K}) = g(K) + \ell g(K_C). \quad (13.1)$$

However, since  $K_C$  is a fibred knot, we see:

$$g(K_C) = \frac{1}{2} \deg \Delta_{K_C}(t). \quad (13.2)$$

With the fact that  $g(K) = \frac{1}{2} \deg \Delta_K(t)$  and Seifert's theorem [3, Proposition 8.23 (b)], we obtain

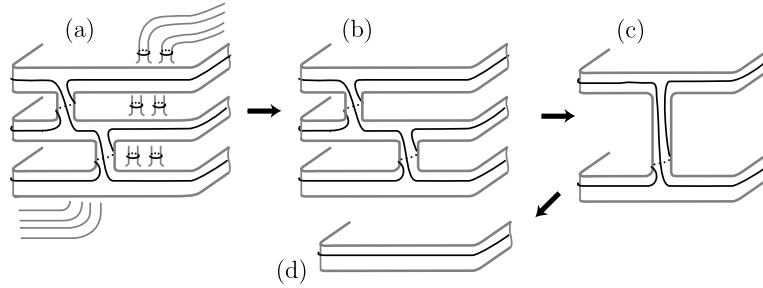
$$2g(\widehat{K}) = \deg \Delta_K(t) + \ell \deg \Delta_{K_C}(t) = \deg \Delta_{\widehat{K}}(t). \quad (13.3)$$

This proves (1).

Next, we prove (2). Suppose  $\Delta_{K(2\alpha, \beta|r)}(t)$  is monic. Then construct a fibre surface  $F$  for  $K = K(2\alpha, \beta|r)$  as in Section 9. As we did in the proof of (1), construct a minimal genus Seifert surface  $\widehat{F}$  for  $\widehat{K}$ . If we needed compressions to obtain  $F$  from the surface  $F(r)$ , then we undo the cuts in  $\widehat{F}$  caused by compressions by plumbing Hopf bands. Note this is possible since the plumbings of Hope band in the proof of Theorem 9.1 can be locally done in  $V$ . Denote by  $F'$  the resulting Seifert surface. To complete the proof, it suffices to show that  $F'$  is a fibre surface. Now  $F'$  looks like as in Figure 7.7, where horizontal parts are understood as parallel copies of  $S_C$ . We can remove the bands by deplumbing Hopf bands as we did in the proof of Proposition 8.1, until we have a new Seifert surface  $F''$  'consisting' of the annuli, and  $\ell$  copies of  $S_C$  and  $\ell - 1$  half-twisted band, where each pair of adjacent copies of  $S_C$  is connected by a half-twisted band. We show that  $F''$  is a fibre surface by using sutured manifolds. Let  $(F'' \times I, \partial F'' \times I)$  be the sutured manifold obtained from  $F''$ . Apply a  $C$ -product disk decomposition corresponding to each site of ribbon holes arising from the intersection of the annuli and copies of  $S_C$ . Then we have 1-handles each with a meridional suture. Actually, we have  $\ell(\lambda - \ell)/2$  1-handles. See Figure



13.1 (a). We can remove, by  $C$ -product decompositions, all such 1-handles coming from the annuli (Figure 13.1 (b)). Apply a  $C$ -product decomposition between a pair of parallel copies of  $S_C$ 's. Then the complementary sutured manifold is separated into two pieces: one of which is a product sutured manifold between two copies of  $S_C$  and hence we can disregard it (Figure 13.1 (c)). Repeating this, we have a sutured manifold obtained from  $S_C$  (Figure 13.1 (d)). Since  $S_C$  is a fibre surface, the complementary sutured manifold is a product sutured manifold. Therefore,  $F''$  is a fibre surface. Theorem 13.1 (2) is now proved.  $\square$

Figure 13.1:  $C$ -product decompositions

*Proof of Theorem 13.2.* For any  $r > 0$  and any fibred companion  $K_C$ , we see  $g(\widehat{K}) \geq g(K(2\alpha, \beta|r))$ . However, since  $\ell = 0$ , from the construction of  $F_1$  in Section 7, we have  $g(\widehat{K}) = g(K(2\alpha, \beta|r))$ , and hence (2) follows immediately from Theorem 2.3.  $\square$

**Remark 13.3.** When  $r = 0$ ,  $K(2\alpha, \beta|0)$  is a trivial knot. If we consider the satellite knot  $\widehat{K}$  with  $K(2\alpha, \beta|0)$  as a pattern knot, this satellite knot gives rise to a very interesting problem. As is known to some specialists, even if the pattern and the companion of  $\widehat{K}$  are both fibred,  $\widehat{K}$  may not be fibred [4]. Therefore, the determination of the genus and fibredness of a satellite knot  $\widehat{K}$  with a fibred companion and the pattern  $K(2\alpha, \beta|0)$  is not straightforward. We leave this problem untouched. To the reader who are interested in this problem, we refer to [11].

## 14. Genus one knots

In this final section, we determine torti-rational knots  $K(2\alpha, \beta|r)$  of genus one (Theorem 14.2). Recall that torti-rational knots have tunnel number one. (We denote the tunnel number of  $K$  by  $t(K)$ .)

Recently Scharlemann [18] proved a conjecture by Goda and Teragaito which states that if a knot  $K$  has  $g(K) = t(K) = 1$ , then  $K$  is a 2-bridge knot or satellite knot. Before that, Goda and Teragaito [8] had classified satellite knots  $K$  with  $g(K) = t(K) = 1$ .

**Proposition 14.1.** [8, Proposition 18.1] *Let  $K$  be a satellite knot with  $g(K) = t(K) = 1$ . Then the pattern knot is genus one 2-bridge knot. More precisely, the*

pattern knot is the torti-rational knot  $K(8d, 4d + 1|pq)$  and the companion knot is a torus knot  $T(p, q)$ .

Note that  $(4d+1)/8d = [2, 2d, -2]$ , and hence the pattern knot is a 2-bridge knot whose associated continued fraction is  $[2d, 2pq]$ . In Proposition 14.1, it is evident that the associated 2-bridge link has linking number zero, and that the pattern knot has genus one. Hence Proposition 14.1 immediately follows from Theorem 14.2 below.

In Theorem 14.2, we have a family of torti-rational knots  $K$  with  $g(K) = t(K) = 1$ , which, at a glance, looked like counter-examples to the Goda-Teragaito conjecture (See Example 14.6).

**Theorem 14.2.** *A torti-rational knot  $K = K(2\alpha, \beta|r)$  with  $\ell = \ell k B(2\alpha, \beta) \geq 0, r > 0$  has  $g(K) = 1$  if and only if one of the following is satisfied:*

*Case A:  $\ell = 0$ .*

*A1:  $\beta/2\alpha = \pm[2, 2d, -2], d \neq 0$  and  $r$  is arbitrary.*

*Case B:  $\ell > 0$ .*

*B1:  $\beta/2\alpha = [2, 2, 2], r = 2$*

*B2:  $\beta/2\alpha = [2, 2, 2, 2, 2], r = 1$*

*B3:  $\beta/2\alpha = [2, 2d, 2], d \neq 1, r = 1$ . Note possibly,  $d = 0$ .*

*B4:  $\beta/2\alpha = \pm \underbrace{[2, 2, \dots, 2]_{2a+1}}_{2a+1}, \underbrace{[-2, -2, \dots, -2]_{2a-1}}_{2a-1}, \text{ or } \pm \underbrace{[-2, -2, \dots, -2]_{2a-1}}_{2a-1}, \underbrace{[2b, 2, 2, \dots, 2]_{2a+1}}_{2a+1},$*

*$a \geq 1, b \neq 0, r = 1$ .*

**Remark 14.3.** In Theorem 14.2,  $K = K(2\alpha, \beta|r)$  in A1, or in B4 with  $a = 1$ , is a 2-bridge knot. If  $K$  is in B1 or B2, then  $K$  is a trefoil knot. If  $K$  is in B3, then  $K$  is a twist knot. If  $K$  is in B4 with  $a \geq 2$ , then  $K$  is a satellite knot with its companion a torus knot  $T(a, a + 1)$  and the pattern knot  $B(4ab(a + 1) - 1, 2a(a + 1))$ , where the Alexander polynomial is  $\Delta_{K(1)}(t) = ab(a + 1)(t - 1)^2 + t$ .

As a direct consequence, we have:

**Corollary 14.4.** *A torti-rational knot  $K$  of genus one is a satellite knot if and only if  $K$  is in Case B4 with  $a \geq 2$  of Theorem 14.2.*

**Remark 14.5.** (1) Our knots in B4 with  $a \geq 2$  has genus one and hence is prime and not a cable knot. Therefore, they are negative examples to the question posed in [1].

(2) We can also prove that if  $\beta/2\alpha = [4, \pm 2, \pm 4]$ , or  $[4, \pm 3, \pm 4]$ , then  $K(2\alpha, \beta|\pm 1)$  cannot be a prime satellite knot (c.f. [7, Theorem 1.6 (2)]).

**Example 14.6.** Consider a 2-bridge link  $B(46, 39)$ , where  $39/46 = [2, 2, 2, 2, 2, -2, -2, -2]$ . (This is the case  $(a, b) = (2, 1)$  in Case B4). The diagram of  $K(r), r = 1$ , is depicted in Figure 14.1(a) together with a compressible Seifert surface  $F$ .

Compressing  $F$  three times, we obtain a minimal genus Seifert surface  $F'$  of genus 1, isotopic to those depicted in Figures 14.1 (b), (c) and (d). Then we see that  $F'$  is a Seifert surface for the satellite knot with its companion  $T(2, 3)$  and the pattern knot a 2-bridge knot  $B(23, 12)$ .

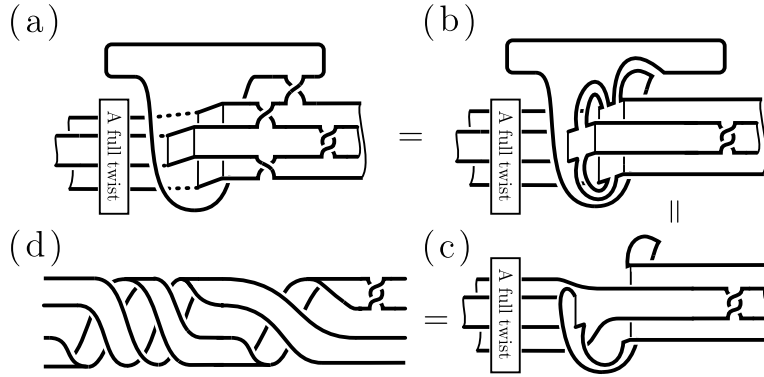


Figure 14.1:  $g(K(46, 39|1)) = 1$

*Proof of Theorem 14.2.* Case A: By (10.1),  $g(K) = 1$  if and only if  $\beta/2\alpha = \pm[2, 2d, -2], d \neq 0$ . Therefore,  $K$  is a 2-bridge knot associated to the continued fraction  $[\pm 2d, 2r]$ .

Case B: By Proposition 6.3, we see

$$2(\lambda - 1 - \rho) + (\lambda - 1)(r - 1)\ell + (\lambda - 2)(\ell - 1) = 2. \quad (14.1)$$

Since each term of the LHS in (14.1) is non-negative, we have two cases: (i)  $\lambda - 1 - \rho = 0$  and (ii)  $\lambda - 1 - \rho = 1$ . For case (i), as is seen in the proof of Proposition 6.6, we have  $(2\alpha, \beta) = (2\lambda, 2\lambda - 1)$  and  $\lambda = \ell$ , and hence, either (a)  $\lambda = \ell = 2$  and  $r = 2$ , or (b)  $\lambda = \ell = 3$  and  $r = 1$ . Therefore, we have either B1 or B2. For case (ii), we have either (c)  $\lambda = 2$  and hence  $r = 1$  and  $\rho = 0$ , or (d)  $\lambda > 2, \ell = r = 1$  and hence  $\rho = \lambda - 2$ . Therefore, we have B3 or B4. This proves Theorem 14.2.  $\square$

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